PS2 Solutions

1. The states are labeled by the number of open links s, where
$$\varepsilon_s = s \cdot \varepsilon \quad \text{and} \quad \Omega_s = 1$$

$$Z = \sum_s \Omega_s e^{-\varepsilon_s/kT} = \sum_{s=0}^{N} e^{-s\varepsilon/kT} = \frac{1-e^{-(N+1)\varepsilon/kT}}{1-e^{-\varepsilon/kT}}$$

$$\langle s \rangle = \text{avg. \# of open links} = \frac{1}{\varepsilon}\langle E \rangle = \frac{1}{\varepsilon} kT^2 \frac{\partial}{\partial T} \log Z$$

$= $ mess, in general

For $\varepsilon \gg kT$ $\quad Z \simeq 1 + e^{-\varepsilon/kT}$

$$\langle s \rangle = \frac{1}{\varepsilon} \cdot \frac{\varepsilon e^{-\varepsilon/kT}}{1+e^{-\varepsilon/kT}} = \frac{1}{e^{\varepsilon/kT}+1} < 1$$

why? it costs too much energy to hold a link open

For $\varepsilon \ll kT$
$$Z \simeq \frac{\frac{N+1}{kT}\varepsilon - \frac{1}{2}\left(\frac{N+1}{kT}\varepsilon\right)^2 + \cdots}{\frac{\varepsilon}{kT} - \frac{1}{2}\left(\frac{\varepsilon}{kT}\right)^2 + \cdots}$$

$$= (N+1)\frac{1 - \frac{1}{2}\frac{N+1}{kT}\varepsilon + \cdots}{1 - \frac{1}{2}\frac{\varepsilon}{kT} + \cdots}$$

$$\simeq (N+1)\left(1 - \frac{N\varepsilon}{2kT} + \cdots\right)$$

$$\log Z = \log(N+1) - \frac{N\varepsilon}{2kT}$$

$$\langle s \rangle = \frac{1}{\varepsilon} \cdot kT^2 \cdot \frac{N\varepsilon}{2kT^2} = \frac{N}{2}$$! open + closed states are equally likely

2. $S = \frac{E-F}{T}$ from the thermo. definition of F

$$= \frac{1}{T}\sum_s \varepsilon_s p_s + k_B \log Z$$

$$= -k_B \sum_s p_s \underbrace{\left(\frac{-\varepsilon_s}{k_B T} - \log Z\right)}_{= \log p_s} \quad \text{since } \sum_s p_s = 1$$

$$= -k_B \sum_s p_s \log p_s$$

This formula also works for the micro-canonical ensemble where $p_s = \frac{1}{\Omega(E)}$

$$-k_B \sum_s p_s \log p_s \to k_B \log \Omega(E) \sum_s p_s = k_B \log \Omega(E).$$

2. $S = \lambda V^{1/2} (NE)^{1/4}$

(a) $T^{-1} = \left(\frac{\partial S}{\partial E}\right)_{V,N} = +\frac{1}{4}\lambda V^{1/2} N^{1/4} E^{-3/4} = S/4E$

$p = T\left(\frac{\partial S}{\partial V}\right)_{E,N} = T \cdot \frac{1}{2}\lambda V^{-1/2}(NE)^{1/4} = TS/2V$

↳ from $dE = TdS - p\,dV \rightarrow dS = \frac{1}{T}dE + \frac{p}{T}dV$.

(b) In equilibrium $T_1 = T_2$ so

$V_1^{1/2} N_1^{1/4} E_1^{-3/4} = V_2^{1/2} N_2^{1/4} E_2^{-3/4}$

or $\left(\frac{E_2}{E_1}\right)^{3/4} = \left(\frac{V_2}{V_1}\right)^{1/2}\left(\frac{N_2}{N_1}\right)^{1/4} = 1 \cdot \left(\frac{2/3}{1/3}\right)^{\frac{1}{4}}$

so $E_2/E_1 = 2^{1/3}$

(c) Here $T_1 = T_2$ and $p_1 = p_2$

T: $V_1^{1/2}\left(\frac{1}{3}\right)^{1/4} E_1^{-3/4} = V_2^{1/2}\left(2/3\right)^{1/4} E_2^{-3/4}$

p: $V_1^{-\frac{1}{2}}\left(1/3\right)^{1/4} E_1^{1/4} = V_2^{-\frac{1}{2}}\left(2/3\right)^{1/4} E_2^{1/4}$

divide: $V_1/E_1 = V_2/E_2$

multiply: $(1/3)^{\frac{1}{2}} E_1^{-\frac{1}{2}} = (2/3)^{1/2} E_2^{-1/2}$

or $E_2/E_1 = 2$