PS 9 Solutions

1. $H(p,z) = p^2/2m + mgz$.

If new Hamiltonian is P, simplest way to get this is a time-indep transformation,

$$P = \frac{p^2}{2m} + mgz, \quad K(PQ) = H(p(PQ), z(PQ))$$

Try to get $z = \frac{1}{mg}\left(P - p^2/2m\right)$ to fall out of the transformation:

$$z = -\frac{\partial F_4(p,P)}{\partial p}$$ will do, + gives

$$F_4 = \frac{-1}{mg}\left(Pp - p^3/6m\right), \quad Q = \frac{\partial F_4}{\partial P} = -p/mg.$$

$$K = P$$

Then eq of motion are $\dot{P} = -\frac{\partial K}{\partial Q} = 0$

$$\dot{Q} = \frac{\partial K}{\partial P} = 1$$

so $Q = t$, $P = \text{constant} = E$

$$p = -mgQ = -mgt$$

$$z = \frac{1}{mg}\left(P - p^2/2m\right) = \frac{1}{mg}\left(E - \tfrac{1}{2}m^2g^2t^2\right)$$

2a) $Q = 1/q \qquad P = pq^2$

$$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{-1}{q^2}\cdot q^2 - 0 = -1$$

$\therefore$ no

b) $Q = 1/p \qquad P = qp^2$

$$[Q,P] = -\left(\frac{-1}{p^2}\right)(p^2) = +1 \quad \therefore \text{yes}$$

$$F_4 = P/p \;:\; q = -\frac{\partial F_4}{\partial p} = \frac{P}{p^2}, \quad Q = \frac{\partial F_4}{\partial P} = \frac{1}{p} \quad \checkmark$$

c) $Q = C(p + i\omega q) \qquad P = C(p - i\omega q)$

$$[Q,P] = (i\omega C)(C) - (C)(-i\omega C) = 2i\omega C^2$$

canonical if $C = \dfrac{1}{\sqrt{2i\omega}}$

d) $Q = q\cos\alpha - p\sin\alpha \qquad P = q\sin\alpha + p\cos\alpha$

$$[Q,P] = \cos^2\alpha + \sin^2\alpha = +1. \quad \therefore \text{yes}$$

To get the GF in c, d, consider

$$\left.\begin{aligned} Q &= \alpha p + \beta q \\ P &= \gamma p + \delta q \end{aligned}\right\} \quad [Q,P] = \beta\gamma - \alpha\delta \equiv 1$$

for appropriate $\alpha, \beta, \gamma, \delta$

rewrite: $p = \frac{Q - \beta q}{\alpha}$, $P = \gamma\left(\frac{Q-\beta q}{\alpha}\right) + \delta q = \frac{\gamma Q - q}{\alpha}$

use $F_1(q, Q)$: $p = \frac{\partial F_1}{\partial q}$ $P = -\frac{\partial F_1}{\partial Q}$

$$\rightarrow \quad \frac{\partial F_1}{\partial q} = \frac{Q - \beta q}{\alpha} \rightarrow F_1 = \frac{1}{\alpha}\left(Qq - \frac{1}{2}\beta q^2\right) + \varphi(Q)$$

$$\frac{\partial F_1}{\partial Q} = -\frac{\gamma Q - q}{\alpha} \rightarrow F_1 = \frac{1}{\alpha}\left(-\frac{\gamma Q^2}{2} + qQ\right) + \psi(q)$$

+ 1 these are the same F_1, need

$$\varphi(Q) = -\gamma Q^2/2\alpha, \quad \psi(q) = -\beta q^2/2\alpha$$

$$+ \quad F_1 = Qq/\alpha - (\beta q^2 + \gamma Q^2)/2\alpha$$

So case (c): $F_1 = (2Qq - \iota w q^2 + CQ^2)/2C$

" (d): $F_1 = -Qq \csc\alpha + \frac{1}{2}\cot\alpha\,(q^2 + Q^2)$

Last case has a problem when $\alpha = \pi$; instead

use $F_2(q, P) = qP\sec\alpha - \frac{1}{2}\tan\alpha\,(q^2 + P^2)$

4 a) $\frac{d\underline{A}}{dt} = [\underline{A}, H] = [\underline{p}\times\underline{L} - mk\hat{r},\ p^2/2m - k/r]$

$$= -k[\underline{p}, 1/r]\times\underline{L} - \frac{k}{2}[\underline{r}/r, p^2]$$

since $[\underline{L}, \text{scalar}] = [r_i, r_j] = [p_i, p_j] = 0$

$$= k\frac{\partial}{\partial \underline{r}}(1/r)\times\underline{L} - k\,\underline{p}\cdot\frac{\partial}{\partial\underline{r}}(\underline{r}/r)$$

$$= -k(\underline{r}/r^3)\times(\underline{r}\times\underline{p}) - k\underline{p}\cdot(\underline{\underline{1}}/r - \underline{r}\,\underline{r}/r^3)$$

$$= -\frac{k}{r^3}[\underline{r}(\underline{r}\cdot\underline{p}) - \underline{p}\,r^2] - k[\underline{p}/r - \underline{r}\,\frac{\underline{p}\cdot\underline{r}}{r^3}] = 0$$

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3. Note that $[q,p]=1 \Rightarrow [q,p^n] = np^{n-1}$

(proof by induction) and then

$$[q, f(p)] = [q, \sum_{n=0}^{\infty} \frac{f_n}{n!}p^n] = \sum_{n=0}^{\infty} \frac{f_n}{n!}\, np^{n-1}$$

$$= f'(p)$$

Similarly $[p, g(q)] = -g'(q)$

Here, $u(q,p,t) = \log(p + im\omega q) - i\omega t$

$$H(q,p) = p^2/2m + \frac{1}{2}m\omega^2 q^2$$

so the various terms are

$$[u, p^2] = 2p[u,p] = 2p\frac{\partial u}{\partial q} = \frac{2imwp}{p+imwq}$$

$$[u, q^2] = 2q[u,q] = -2q\frac{\partial u}{\partial p} = \frac{-2q}{p+imwq}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + [u, H]$$

$$= -iw + \underbrace{\frac{iwp}{p+imwq} + \frac{-imw^2 q}{p+imwq}}_{+iw} = 0.$$

If $u(q,p,t) = C = \log(p+iwmq) - iwt$

then $p + iwmq = e^{C+iwt} \longrightarrow Ae^{i(wt+\varphi)}$

$C = Ae^{i\varphi}$

so $p = A\cos(wt+\varphi)$ $\quad q = \frac{A}{mw}\sin(wt+\varphi)$

Invariance of H under changes of u corresponds to invariance under changes in the wave amplitude & phase, which in turn corresponds to invariance under variations in the initial conditions.