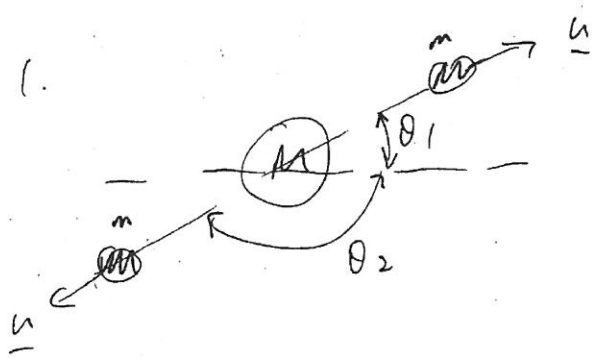
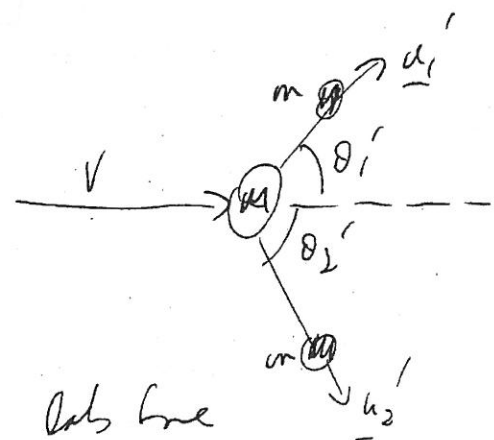
Problem Set 8

1.



CM frame



Lab frame

In CM, energy conservation $\rightarrow$ $Mc^2 = 2mc^2\gamma_{\underline{u}}$

so $|\underline{u}| = c\sqrt{1-4m^2/M^2}$.

Velocities transform as

$$u_{1x}' = \frac{u_{1x}+V}{1+u_{1x}V/c^2}, \quad u_{1y}' = \frac{u_{1y}}{\gamma_V(1+u_{1x}V/c^2)}$$

and $u_{1x} = u\cos\theta_1$, $u_{1y} = u\sin\theta_1$, etc.

$$\text{so} \quad \tan\theta_1' = \frac{u_{1y}'}{u_{1x}'} = \frac{u_{1y}}{\gamma_V(u_{1x}+V)} = \frac{u\sin\theta_1}{\gamma(u\cos\theta_1+V)}$$

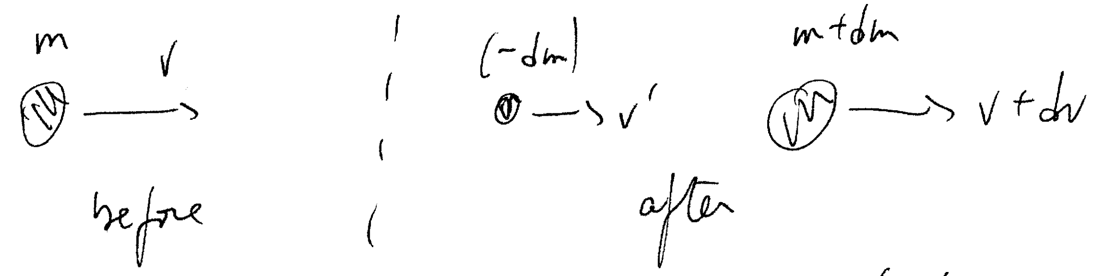
also $\theta_2 = \pi-\theta_1$ so $\tan\theta_2' = \dfrac{u\sin\theta_{\cancel{2}1}}{(-u\cos\theta_1+V)\gamma}$

The separation angle in the lab frame is $\Theta = \theta_1'+\theta_2'$

$$\text{so} \quad \tan\Theta = \frac{\tan\theta_1' + \tan\theta_2'}{1+\tan\theta_1'\tan\theta_2'} = \cdots$$

Note the only variable parameter is θ_1.

2.



$(dm<0)$

u is the relative velocity so $v' = \dfrac{v-u}{1-vu/c^2}$.

Conservation of energy + 3 momentum give

$$\begin{cases} \dfrac{mc^2}{\sqrt{1-v^2/c^2}} = \dfrac{(-dm)c^2}{\sqrt{1-v'^2/c^2}} + \dfrac{(m+dm)c^2}{\sqrt{1-(v+dv)^2/c^2}} \\[2ex] \dfrac{mv}{\sqrt{1-v^2/c^2}} = \dfrac{(-dm)v'}{\sqrt{1-v'^2/c^2}} + \dfrac{(m+dm)(v+dv)}{\sqrt{1-(v+dv)^2/c^2}} \end{cases}$$

or

$$\begin{cases} 0 \simeq dm\left[1-\sqrt{\dfrac{1-v^2/c^2}{1-v'^2/c^2}}\right] + dv\left[\dfrac{mv/c^2}{1-v^2/c^2}\right] \\[2ex] 0 \simeq dm\left[-v'\sqrt{\dfrac{1-v^2/c^2}{1-v'^2/c^2}}+v\right] + dv\left[m+\dfrac{mv^2/c^2}{1-v^2/c^2}\right] \end{cases}$$

Subtract:

$$0 = (v-v')\,dm + \underbrace{\left[1+\frac{v(v-v')/c^2}{1-v^2/c^2}\right]}_{=\,1/1-uv/c^2} m\,dv$$

since $v-v' = u\,\dfrac{1-v^2/c^2}{1-uv/c^2}$

$$\rightarrow\ m\frac{dv}{dm} = -u\left(1-v^2/c^2\right)$$

$$\rightarrow\ \frac{v}{c} = \frac{1-(m/m_0)^{2u/c}}{1+(m/m_0)^{2u/c}} = \tanh\left[\frac{u}{c}\log\frac{m_0}{m}\right]$$

3. $H = \frac{1}{2m}(\underline{p} - q\underline{A})^2 + q\varphi$ for a particle in an EM field.

Here $\varphi = 0$, $\underline{B} = \text{const} \to B\hat{z}$, so $\underline{A} = \frac{B}{2}(-y, x, 0)$ say.

$$\to H = \frac{1}{2m}\left[\left(p_x + \frac{qB}{2}y\right)^2 + \left(p_y - \frac{qB}{2}x\right)^2 + p_z^2\right]$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}, \quad \dot{p}_z = -\frac{\partial H}{\partial z} = 0$$

$$\text{so } p_z = \text{const.} \equiv m\dot{z}_0, \quad z(t) = z_0 + \dot{z}_0 t.$$

$$\dot{x} = \frac{1}{m}\left(p_x + \frac{qB}{2}y\right) \qquad \dot{p}_x = \frac{qB}{2}\left(p_y - \frac{qB}{2}x\right)$$

$$\dot{y} = \frac{1}{m}\left(p_y - \frac{qB}{2}x\right) \qquad \dot{p}_y = \frac{qB}{2}\left(p_x + \frac{qB}{2}y\right)$$

Notice $\dot{p}_y + \frac{qB}{2}\dot{x} = 0$ so $p_y + \frac{qB}{2}x = \text{const.} \equiv qBK_1$

$\dot{p}_x - \frac{qB}{2}\dot{y} = 0$ so $p_x - \frac{qB}{2}y = \text{const} \equiv -qBK_2$

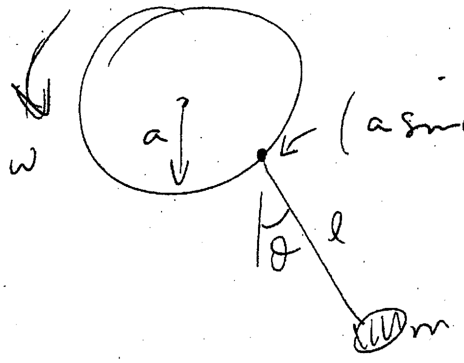
$$\to \frac{d}{dt}(x - K_1) = \omega(y - K_2) \qquad \omega \equiv \frac{qB}{m}$$

$$\frac{d}{dt}(y - K_2) = -\omega(x - K_1)$$

$$\to \frac{d^2}{dt^2}(x - K_1) = -\omega^2(x - K_1) \quad \text{so} \begin{cases} x = K_1 + A\sin(\omega t + \varphi) \\ y = K_2 + A\cos(\omega t + \varphi) \\ p_x, p_y = \ldots \end{cases}$$

Fit K_1, K_2, A, φ to initial data.

4.



$$\underline{x} = \left(a\sin\omega t + l\sin\theta,\ a\cos\omega t + l\cos\theta\right)$$

$$T = \frac{m}{2}\dot{\underline{x}}^2, \quad V = -mgy$$

$$\rightarrow L = \frac{m}{2}\left[l^2\dot\theta^2 + a^2\omega^2 + 2al\omega\dot\theta\cos(\theta-\omega t)\right] + mg\left(a\cos\omega t + l\cos\theta\right)$$

$$p = \frac{\partial L}{\partial\dot\theta} = m\left[l^2\dot\theta + al\omega\cos(\theta-\omega t)\right]$$

$$H = p\dot\theta - L = \frac{1}{2ml^2}\left[p - mal\omega\cos(\theta-\omega t)\right]^2 - mg\left(l\cos\theta + a\cos\omega t\right) - ma^2\omega^2/2$$

$$\dot\theta = \frac{\partial H}{\partial p} = \frac{1}{ml^2}\left[p - mal\omega\cos(\theta-\omega t)\right]$$

$$\dot p = -\frac{\partial H}{\partial\theta} = -\frac{a\omega}{l}\left[p - mal\omega\cos(\theta-\omega t)\right]\sin(\theta-\omega t) - mgl\sin\theta$$

notice H is fn. of time, not conserved or $= E$

p = ang. momentum about point of suspension