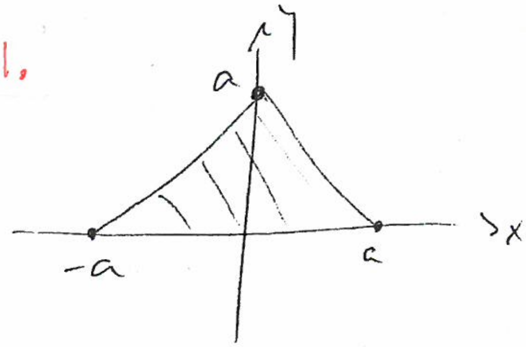
Problem Set 5 Solutions

1.



Uniform mass density ρ

Find $\underline{\underline{I}}$ wrt origin then translate

$$M = 2\int_0^a dx \int_0^{a-x} dy \cdot \rho = \frac{1}{2}\rho a^2$$

$$x_{CM} = z_{CM} = 0, \quad y_{CM} = \frac{1}{M}\int_0^a dx \int_0^{a-x} dy \cdot \rho y = \frac{a}{3}.$$

$$I_{xx} = 2\int_0^a dx \int_0^{a-x} dy \cdot \rho y^2 = \frac{1}{6} Ma^2$$

$$I_{yy} = \quad \text{"} \quad \text{"} \quad \rho x^2 = \frac{1}{6} Ma^2$$

$$I_{xy} = \quad \text{"} \quad \text{"} \quad \rho xy = 0 = I_{xz} = I_{yz}$$

$$I_{zz} = \quad \text{"} \quad \text{"} \quad \rho(x^2+y^2) = \frac{1}{3} Ma^2$$

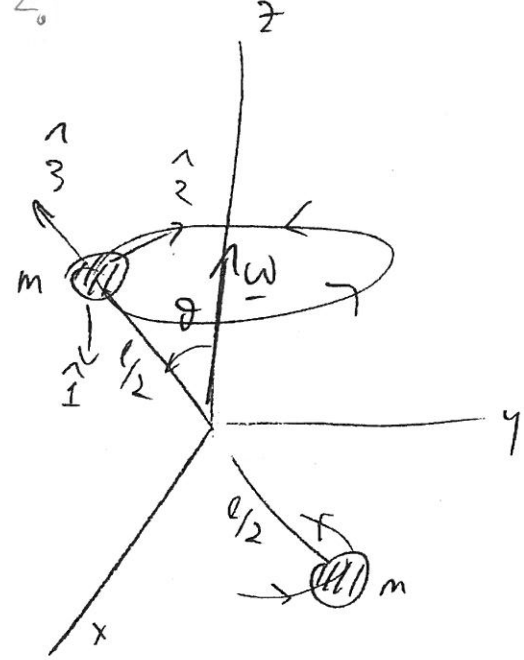
Parallel-axis thm:

$$\underline{\underline{I}}_{CM} = \underline{\underline{I}} - M\left(\underline{r}^2 \underline{\underline{1}} - \underline{r}\,\underline{r}\right) \qquad \underline{r} = \text{vector from } \underline{0} \text{ to CM} = (0, a/3, 0)$$

$$= \begin{pmatrix} \frac{1}{6}Ma^2 & & 0 \\ & \frac{1}{6}Ma^2 & \\ 0 & & \frac{1}{3}Ma^2 \end{pmatrix} - M\begin{pmatrix} a^2/9 & & 0 \\ & 0 & \\ 0 & & a^2/9 \end{pmatrix}.$$

$$= \begin{pmatrix} \frac{1}{18}Ma^2 & & 0 \\ & \frac{1}{6}Ma^2 & \\ 0 & & \frac{2}{9}Ma^2 \end{pmatrix}$$

2.



(x, y, z) = space frame (inertial) with $\underline{\omega} = \omega \hat{z}$

$(1, 2, 3)$ = body frame with

$\hat{3}$ = bar axis

$\hat{2}$: in plane of $\underline{\omega}$ + $\hat{3}$

$\hat{1}$: $\perp$ plane of "

(This choice corresponds to an Euler angle rotation off the z-axis + is convenient.)

Then $\underline{\omega}_{body} = (0, \omega \sin\theta, \omega\cos\theta)$

$\underline{\underline{I}}_{body} = \text{diag}(I, I, 0)$, $I = 2 \times m \times \left(\frac{l}{2}\right)^2 = \frac{1}{2} ml^2$

$\underline{L}_{body} = (0, I\omega\sin\theta, 0)$

a) $\underline{N} = \underset{\overset{\shortparallel}{0}}{\dot{\underline{L}}} + \underline{\omega} \times \underline{L}$ in body frame

$N_1 = \omega_2 L_3 - \omega_3 L_2 = -I\omega^2 \sin\theta\cos\theta$

$N_2 = N_3 = 0$

b) Method 1 : use the Euler rotation (4.47) for $\underline{A}^T$ with $\psi = 0$, $\varphi = \omega t$ to transform $\underline{L}_{body}$ to $\underline{L}_{space}$

$\rightarrow$ $L_x = -\cos\theta \sin\omega t \cdot L_2$

$L_y = \cos\theta\cos\omega t \cdot L_2$, $L_z = \sin\theta \cdot L_2$

In the space frame,

$$\underline{N} = \frac{d\underline{L}}{dt} = \underbrace{I\omega^2 \sin\theta\cos\theta}_{|\underline{N}|} \big(\underbrace{-\hat{x}\cos\omega t - \hat{y}\sin\omega t}_{\text{rotating unit vector in } x\text{-}y \text{ plane}}\big)$$

which agrees: the fixed vector $\underline{N}$ in the body frame rotates with frequency ω as seen in the space frame.

Method 2: In space coordinates, the trajectories of the ~~M~~ masses m is just

$$* \quad \underline{r}_{1,2}(t) = \left(\pm\frac{l}{2}\sin\theta\sin\omega t,\ \mp\frac{l}{2}\sin\theta\cos\omega t,\ \pm\frac{l}{2}\cos\theta\right)$$

so

$$\underline{L} = \sum_{i=1}^{2} \underline{r}_i \times (m\dot{\underline{r}}_i) = \cdots$$

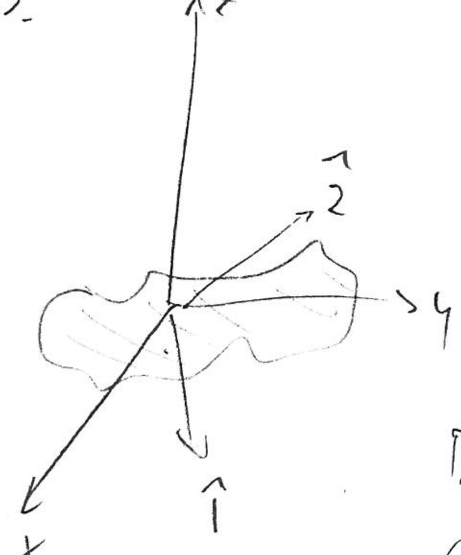
$$= \frac{ml^2\omega}{2}\sin\theta\cos\theta\left[-\hat{x}\sin\omega t + \hat{y}\cos\omega t\right]$$

& $\underline{N} = \dot{\underline{L}}$ = same thing.

* The "phase" of $\underline{v}_{1,2}(t)$ (i.e., choice of $\pm\sin\omega t$, $\mp\cos\omega t$) is chosen so that at $t=0$,

$$\underline{r}_{1,2}(t=0)\Big|_{\text{body}} = \underline{\underline{A}}^T\, \underline{r}_{1,2}(t=0)\Big|_{\text{space}} = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}$$

3.



Each mass point has $z = 0$ so

$$\underline{\underline{I}} = \begin{pmatrix} \sum m y^2 & -\sum m x y & 0 \\ -\sum m x y & \sum m x^2 & 0 \\ 0 & 0 & \sum m (x^2+y^2) \end{pmatrix}$$

Diagonalize $\underline{\underline{I}}$ in the x-y subspace; rotate $(\hat{x}, \hat{y})$ to $(\hat{1}, \hat{2})$ in the body system, where

$$\underline{\underline{I}}\Big|_{\text{principal axis}} = \begin{pmatrix} \sum m \hat{y}^2 & & 0 \\ & \sum m \hat{x}^2 & \\ 0 & & \sum m (x^2+y^2) \end{pmatrix} = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix}$$

Then note $x^2+y^2 = \hat{x}^2 + \hat{y}^2$ under rotation, so $I_1 + I_2 = I_3$.

(b) The values $I_1 = \mu^2 - 1$, $I_2 = \mu^2 + 1$, $I_3 = 2\mu^2$ are consistent with (a). The torque-free Euler eqs are

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 \rightarrow \dot{\omega}_1 = -\omega_2\omega_3$$

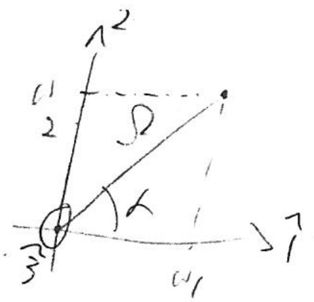
$$I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 \rightarrow \dot{\omega}_2 = +\omega_3\omega_1$$

$$I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 \rightarrow \dot{\omega}_3 = \frac{-1}{\mu^2}\omega_1\omega_2$$

so $\frac{d}{dt}(\omega_1^2 + \omega_2^2) = 2(\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2) = 0$

and $\sqrt{\omega_1^2 + \omega_2^2} = \Omega$, a constant.

(c) let $\tan\alpha = \dfrac{\omega_2}{\omega_1}$:



so $\omega_1 = \Omega\cos\alpha$, $\omega_2 = \Omega\sin\alpha$

+ take $\dfrac{d}{dt}$:

$$\left(\sec^2\alpha\right)\frac{d\alpha}{dt} = \frac{\dot\omega_2}{\omega_1} - \frac{\dot\omega_1\omega_2}{\omega_1^2} = \omega_3 + \frac{\omega_3\omega_2^2}{\omega_1^2}$$

$$= \left(\Omega/\omega_1\right)^2 \rightarrow \dot\alpha = \left(\frac{\omega_1}{\Omega}\right)^2 \omega_3 \frac{\omega_1^2+\omega_2^2}{\omega_1^2} = \omega_3$$

Then $\ddot\alpha = -\dfrac{1}{\mu^2}\omega_1\omega_2 = -\dfrac{\Omega^2}{\mu^2}\sin\alpha\cos\alpha$

At $t=0$, $\underline{\omega} = \mu N\hat{e}_1 + N\hat{e}_2$ so $\boxed{\Omega = \mu N}$ and

$$\boxed{\ddot\alpha + N^2\sin\alpha\cos\alpha = 0}$$

To solve this, let $\beta = \dot\alpha$ so $\ddot\alpha = \dfrac{d\beta}{dt} = \dfrac{d\beta}{d\alpha}\dfrac{d\alpha}{dt} = \beta\beta'$

Integrate : $\frac{1}{2}\beta^2 - \frac{N^2}{2}\cos^2\alpha = C$

+ at $t=0$: $\alpha = \tan^{-1}\dfrac{\omega_2}{\omega_1} = 0$, $\beta = \omega_3 = N \Rightarrow C = 0$

So $\dfrac{d\alpha}{dt} = N\cos\alpha$ or $\dfrac{1}{2}\log\dfrac{1+\sin\alpha}{1-\sin\alpha} = Nt + \underbrace{C'}_{0}$

so $\sin\alpha = \tanh Nt$, $\cos\alpha = \operatorname{sech} Nt$

and $\underline{\omega}(t) = (\mu N\operatorname{sech} Nt)\hat{e}_1 + (\mu N\tanh Nt)\hat{e}_2 + (N\operatorname{sech} Nt)\hat{e}_3$.