PS3 Solutions

1a) If L is invariant under $q_i \to q_i + \delta q_i$ then

$$Q = \sum_i p_i \, \delta q_i \text{ is conserved in time } \left(p_i = \frac{\partial L}{\partial \dot{q}_i} \right).$$

Here $\delta r_i = 0$, $\delta \theta_i = \Delta$, $\delta z_i = \alpha \Delta$

so $$Q = \sum_i \left(p_{\theta_i} \Delta + p_{z_i} \alpha \Delta \right) = \underbrace{\sum_i \left(m_i r_i^2 \dot{\theta}_i + m_i \dot{z}_i \right)}_{\text{conserved.}} \Delta$$

b) Note $L = L_x + L_y + L_z = \left(\frac{m}{2} \dot{x}^2 - \frac{k}{2} x^2 \right) + (y) + (z)$ with no coupling between x, y, z. Each term has no explicit time-dependence, so $E_x = \frac{m}{2} \dot{x}^2 + \frac{k}{2} x^2$, E_y & E_z are conserved. Also L is rotation-invariant as a whole, so $\underline{L} = \underline{r} \times \underline{p}$ is conserved

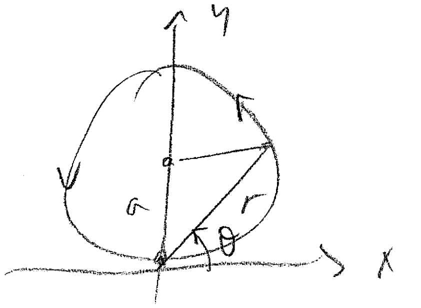
c) $$\dot{C}_{ij} = \frac{1}{2} \left(\underbrace{m \ddot{r}_i}_{-k r_i} \dot{r}_j + \underbrace{m \ddot{r}_j}_{-k r_j} \dot{r}_i + k \dot{r}_i r_j + k r_i \dot{r}_j \right)$$

$$= 0$$

Note $C_{xx} = E_x$, etc., and $C_{xy}^2 = E_x E_y - \frac{k}{4m} L_z$ etc.

so there are 6 independent conserved quantities all together.

2.



a) circular orbit, force center at O, $r = 2a\sin\theta$, $u = 1/r$

$$u''(\theta) + u = \cdots = 8a^2u^3 = -\frac{m}{\ell^2}\frac{d}{du}V\left(\frac{1}{u}\right) = +\frac{m}{\ell^2u^2}\frac{d}{d(1/u)}V(1/u)$$

so $$f(r) = -\frac{dV}{dr} = -\frac{d}{d(1/u)}V(1/u) = -\frac{8a^2\ell^2}{m}\frac{1}{r^5}$$

$$\rightarrow \quad V(r) = -\frac{2a^2\ell^2}{m}\frac{1}{r^4} \quad \text{if } V(\infty) = 0$$

b) $$E = \frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2\right) + V(r) = \frac{m}{2}\left(\left(\frac{dr}{d\theta}\right)^2 + r(\theta)^2\right)\dot\theta^2 + V(r)$$

$$dr/d\theta = 2a\cos\theta; \quad \dot\theta = \ell/mr^2$$

$$E = \frac{m}{2}\left((2a\cos\theta)^2 + (2a\sin\theta)^2\right)\cdot\frac{\ell^2}{m^2r^4} - \frac{2a^2\ell^2}{m}\frac{1}{r^4} = 0.$$

c) $$T = \int dt = \int_0^\pi \frac{d\theta}{\dot\theta} = \int_0^\pi \frac{d\theta}{\ell/mr^2} = \frac{m}{\ell}\int_0^\pi d\theta\,(2a\sin\theta)^2$$

$$= 2a^2\pi m/\ell.$$

d) $$x = r\cos\theta = 2a\sin\theta\cos\theta, \quad y = r\sin\theta = 2a\sin^2\theta$$

$$\dot x = 2a(\cos^2\theta - \sin^2\theta)\dot\theta = 2a\cos 2\theta\cdot\frac{\ell}{mr^2} \underset{\substack{r\to0 \\ \theta = 0 \text{ or } \pi}}{\longrightarrow} \infty$$

$$\dot y = 2a\sin 2\theta\cdot\dot\theta = \frac{\ell}{am}\cot\theta \underset{\theta\to0}{\longrightarrow} \infty$$

3. If $V(r) \to -\frac{k}{r} + \frac{h}{r^2} = -ku + hu^2$ then

$$u''(\theta) + u = -\frac{\mu}{\ell^2}\frac{d}{du}V(1/u) = -\frac{\mu}{\ell^2}(-k + 2hu)$$

or $$u'' + \left(1 + \frac{2h\mu}{\ell^2}\right)u = \frac{\mu k}{\ell^2}$$

or $$\frac{d^2u}{d\hat{\theta}^2} + u = \frac{\mu\hat{k}}{\ell^2}, \quad \left\{\begin{array}{l} \hat{\theta} \equiv \theta\sqrt{1+2h\mu/\ell^2} = \lambda\theta \\ \hat{k} = k/(1+2h\mu/\ell^2) = \lambda^{-2}k \end{array}\right.$$

This is the usual orbit eq, so when $\hat{\theta}$ changes by 2π, $u(\hat{\theta})$ returns to its original value, but θ has shifted by $2\pi/\lambda < 1$ & the orbit seems to have rotated through an angle $2\pi(1-1/\lambda)$. The "precession" angular velocity is

$$\omega = \frac{1}{\tau}2\pi(1-1/\lambda) \qquad \tau = \text{original period}$$

$$\simeq \frac{2\pi\mu h}{\tau\ell^2} \qquad \text{for } |h\mu/\ell^2| \ll 1.$$

etc.

Note that while h/r^2 looks like a change in the centrifugal barrier term $\ell^2/2mr^2$, $\ell = mr^2\dot{\theta}$ is a constant of the motion fixed by the initial conditions.