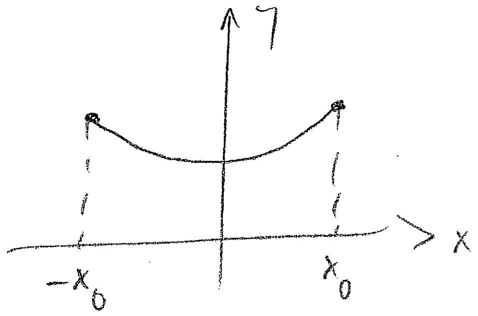
PS 2 Solutions

1. Minimize $V = \int_{-x_0}^{x_0} \rho g\, ds \cdot y = 2\rho g \int_0^{x_0} dx\, y\sqrt{1+y'^2}$



with fixed $l = \int_{-x_0}^{x_0} ds$

$= 2\int_0^{x_0} dx\sqrt{1+y'^2}$

Use a Lagrange multiplier – minimize $V - \lambda l$

or $2\rho g \int_0^{x_0} dx\, \underbrace{(y-\hat{\lambda})\sqrt{1+y'(x)^2}}_{\varphi(y,y')}$ $\quad \hat{\lambda} \equiv \lambda/\rho g$

Since φ does not depend explicitly on x,

$$\varphi - y'\frac{\partial \varphi}{\partial y'} = \text{constant}$$

or $$(y-\hat{\lambda})\left[\sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}}\right] = \frac{y-\hat{\lambda}}{\sqrt{1+y'^2}} = K^2$$

Integrate: $\quad y = \hat{\lambda} + \frac{1}{K}\cosh(Kx + c)$

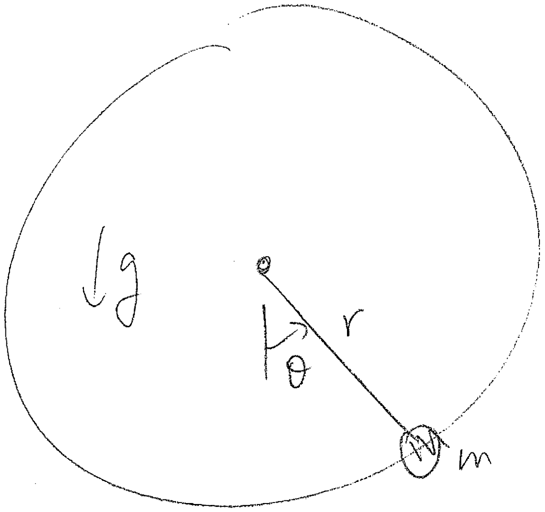
other integration constant

$\hat{\lambda}$ is just a vertical shift

$y(x) = y(-x) \Rightarrow c = 0$

K fixed by l.

2.



$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$

+ constraint $r = L$

as in lecture:
$$\begin{cases} m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = \lambda \\ \frac{d}{dt}\left(mr^2\dot{\theta}\right) + mgr\sin\theta = 0 \end{cases}$$

Impose the constraint:
$$\begin{cases} \lambda = -mL\dot{\theta}^2 - mg\cos\theta \\ mL^2\ddot{\theta} + mgL\sin\theta = 0 \end{cases}$$

Multiply 2nd eqn by $\dot{\theta}$ + integrate (cons. of E):

$$\frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta = E = \frac{1}{2}mL^2\omega^2 - mgL$$

using $\dot{\theta} = \omega$ + $\theta = 0$ at $t = 0$

So $\lambda = -mL\left(\omega^2 - 2g/L + 2g/L\cos\theta\right) - mg\cos\theta$

$= -mL\omega^2 + 2mg - 3mg\cos\theta$

$= 0$ when $\cos\theta^* = \left(2mg - mL\omega^2\right)/3mg$

If $\omega^2 < 2g/L$, $\cos\theta^* > 0$, $\theta^* < \pi/2$

so tension $\to 0$ before mass is horizontal.

Notice $E \leq 0$ here so $\theta \leq \frac{\pi}{2}$ necessarily

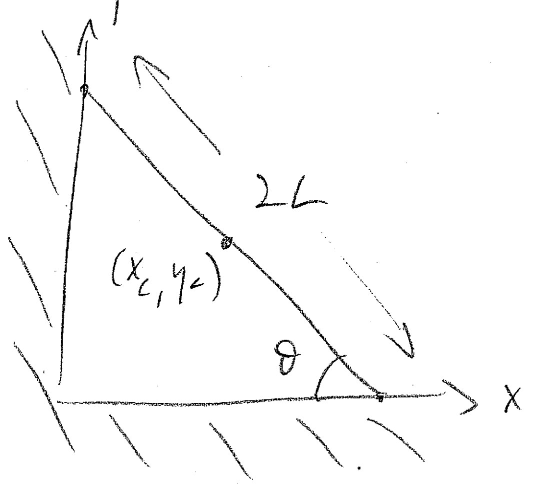
If $2g/L < \omega^2 < 5g/L$ $\quad -1 < \cos\theta^* < 0$

so $\pi/2 < \theta^* < \pi$ + tension $\to 0$ when mass is above the horizontal.

Notice the mass can only reach the top if $E > mgL$ or $\omega^2 > 4g/L$

If $5g/L < \omega^2$ $\quad \cos\theta^* < -1$, θ^* doesn't exist, and the tension is never 0. Here there's enough energy to reach the top + keep rotating.

3.



$$L = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m \left(\dot{x}_c^2 + \dot{y}_c^2 \right) - m g y_c$$

constraints $\begin{cases} x = \text{left end} = 0 \\ y = \text{right end} = 0 \end{cases}$

$$\left.\begin{aligned} x_c &= x + L \cos\theta \\ y_c &= y + L \sin\theta \end{aligned}\right) \text{ in contact } \rightarrow \quad \begin{aligned} \dot{x}_c &= \dot{x} - L \sin\theta \, \dot{\theta} \\ \dot{y}_c &= \dot{y} + L \cos\theta \, \dot{\theta} \end{aligned}$$

$$L \rightarrow \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + 2L\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta) + L^2 \dot{\theta}^2 \right) - mg(y + L\sin\theta)$$

Use constrained Lagrangian method!

$$\left.\begin{aligned} x&: \quad 0 - \frac{d}{dt}\left(m\dot{x} - mL\dot{\theta}\sin\theta \right) = \lambda_x \\ y&: \quad -mg - \frac{d}{dt}\left(m\dot{y} + mL\dot{\theta}\cos\theta \right) = \lambda_y \end{aligned}\right\} \text{ constraint forces}$$

$$\theta: \quad -mgL\cos\theta - \frac{d}{dt}\left(I\dot{\theta} + mL(\dot{y}\cos\theta - \dot{x}\sin\theta) + mL^2\dot{\theta} \right) = 0$$

Impose the constraints $x = y = 0$

$$\rightarrow \quad \lambda_x = mL\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta \right)$$

$$\lambda_y = -mg - mL\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta \right)$$

$$(I + mL^2)\ddot{\theta} + mgL\cos\theta = 0$$

Multiply last eq. by $\dot\theta$ + integrate (or use cons. of E)

$$\to mgL\sin\theta + \frac{1}{2}(I+mL^2)\dot\theta^2 = E = mgL\sin\theta_0$$

if the ladder starts at rest at θ_0.

Substitute for $\ddot\theta$ + $\dot\theta^2$ in F_x:

$$F_x = mL\left[-\frac{mgL\cos\theta}{I+mL^2}\sin\theta + \frac{2mgL(\sin\theta_0-\sin\theta)}{I+mL^2}\cos\theta\right]$$

$$= \frac{m^2gL^2}{I+mL^2}\cos\theta\,(2\sin\theta_0 - 3\sin\theta)$$

$\to 0$ when $\sin\theta^* = \frac{2}{3}\sin\theta_0$

or $\underbrace{2L\sin\theta^*}_{\text{height}} = \frac{2}{3}\,\underbrace{2L\sin\theta_0}_{\text{original height.}}$

After this, there's no x-force so $\dot x_c$ has the same value as at $\theta^* = -L\underbrace{\sin\theta^*}_{\text{known}}\cdot\underbrace{\dot\theta^*}_{\text{find from } E}$

Then $E = \frac{1}{2}I\dot\theta^2 + \frac{1}{2}m\left(\dot x_c^2 + (L\cos\theta\,\dot\theta)^2\right) - mgL\sin\theta$

from which $\dot\theta$ can be found, + integrated in principle.