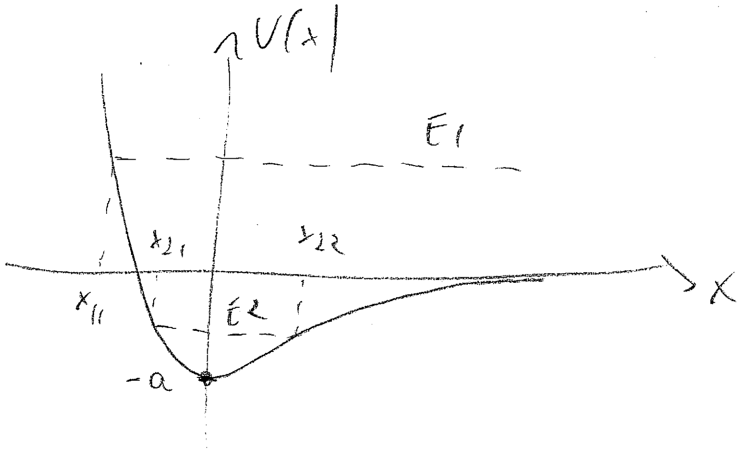
PS1 Solutions

1.



If $E = E_1 > 0$, motion for all $x_{11} < x < \infty$

If $E = E_2 < 0$, motion for $x_{21} < x < x_{22}$

Since $E = \frac{1}{2} m \dot{x}^2 + V(x) = \text{constant}$,

$$t = \int \frac{dx}{\sqrt{\frac{2}{m}\left(E - V(x)\right)}} \quad \text{in general}$$

For the bound case,

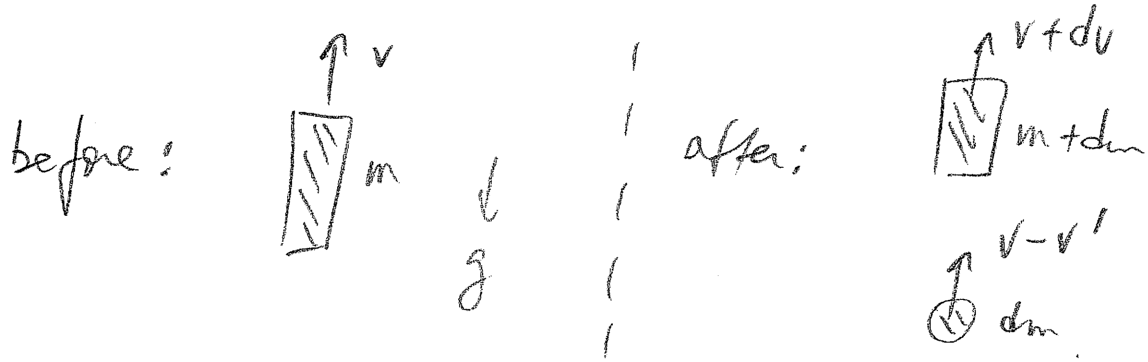
$$T = 2 \int_{x_{21}}^{x_{22}} \frac{dx}{\sqrt{\frac{2}{m}\left(E_2 - V(x)\right)}}$$

$$= \sqrt{2m} \int_{x_{21}}^{x_{22}} \frac{dx}{\sqrt{E_2 - a e^{-2x} + 2a e^{-x}}}$$

$$\xrightarrow[y \equiv e^{x}]{} \sqrt{2m} \int_{y_{21}}^{y_{22}} \frac{dy}{\sqrt{E_2 y^2 + 2ay - a}}$$

etc.

2.



rocket momentum before + gravitational impulse

= rocket momentum after + exhaust gas momentum

$$\cancel{m v} - mg\,dt = (m+dm)(v+dv) + (-dm)(v-v')$$

because $dm < 0$

$$= \cancel{mv} + v\,dm + m\,dv - v\,dm + v'\,dm$$

+ higher order

so $$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg.$$

Integrate! $$v(t) - v_0 = -v' \log\frac{m(t)}{m_0} - gt$$

ok.

3. If $s_i = s_i(q_1 \cdots q_n, t)$ $i = 1, 2, \cdots, n$ then

$$\frac{\partial L}{\partial s_i} = \sum_j \left(\frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial s_i} + \frac{\partial L}{\partial \dot{q}_j} \underbrace{\frac{\partial \dot{q}_j}{\partial s_i}}_{= \frac{d}{dt} \frac{\partial q_j}{\partial s_i}} \right)$$

$$\frac{\partial L}{\partial \dot{s}_i} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \dot{s}_i} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{\partial q_j}{\partial s_i}$$

because $\dot{q}_j = \sum_k \frac{\partial q_j}{\partial s_k} \dot{s}_k + \frac{\partial q_j}{\partial t} \Rightarrow \frac{\partial \dot{q}_j}{\partial \dot{s}_i} = \frac{\partial q_j}{\partial s_i}$

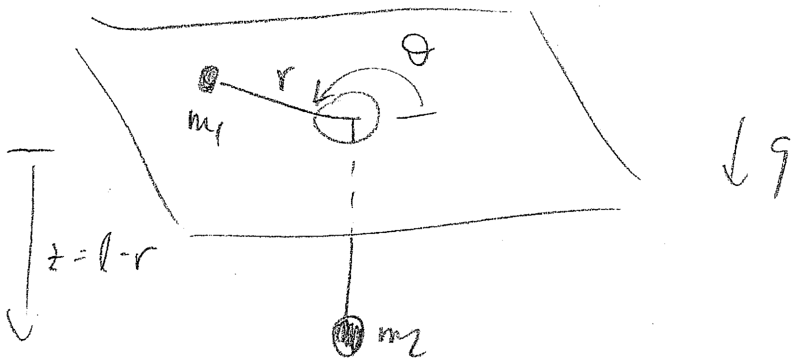
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}_i} = \sum_j \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \frac{\partial q_j}{\partial s_i} + \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \left(\frac{\partial q_j}{\partial s_i} \right) \right)$$

So $$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}_i} - \frac{\partial L}{\partial s_i} = \sum_j \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \right) \frac{\partial q_j}{\partial s_i}$$

$$= 0$$

if L satisfies Lagrange's eqs. in q.

4.



Use polar coordinates in the plane for m_1

use z for m_2 where $z = l - r$

$$T_1 = \tfrac{1}{2} m_1 \dot{\underline{r}}_1^2 = \tfrac{1}{2} m_1 \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right)$$

$$T_2 = \tfrac{1}{2} m_2 \dot{\underline{r}}_2^2 = \tfrac{1}{2} m_2 \dot{z}^2 = \tfrac{1}{2} m_2 \dot{r}^2$$

$$V = -m_2 g z$$

$$L = \tfrac{1}{2}(m_1 + m_2)\dot{r}^2 + \tfrac{1}{2} m_1 r \dot{\theta}^2 + m_2 g (l - r)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 = (m_1 + m_2)\ddot{r} + m_2 g - m_2 r \dot{\theta}^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}\left(m_1 r^2 \dot{\theta} \right)$$

so $L_1 = m_1 r \dot{\theta}^2 = \text{constant}$

$$(m_1 + m_2)\ddot{r} = -m_2 g + \frac{L_1^2}{m_1 r^3}$$

$$E = \tfrac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{L_1^2}{2 m_1 r^2} + m_2 g r = \text{constant.}$$