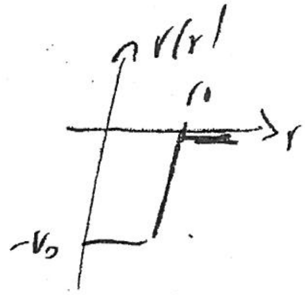
Problem Set 10 Solutions



1. $$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r)$$

H indep of time, θ cyclic

$$\rightarrow \quad E = \frac{1}{2m}\left(p_r^2 + \frac{\alpha_\theta^2}{r^2}\right) + V(r)$$

For action-angle variables want bound motion, p_r real inside the well, so

$$E - \frac{\alpha_\theta^2}{2mr_0^2} + V_0 > 0 \; ; \quad E - \frac{\alpha_\theta^2}{2mr_0^2} < 0$$

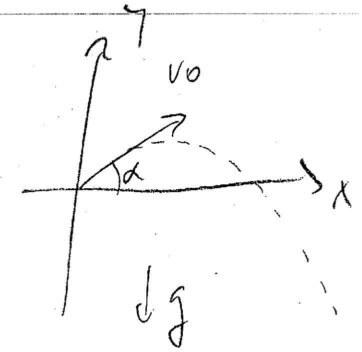
Maximum value of r is r_0,
Minimum " " " is $r_- = \dfrac{|\alpha_\theta|}{\sqrt{2m(E+V_0)}}$

so $$I = \frac{1}{2\pi}\oint p\,dq = \frac{1}{\pi}\int_{r_-}^{r_0} dr \sqrt{2m(E+V_0) - \alpha_\theta^2/r^2}$$

$$\omega^{-1} = \frac{\partial I}{\partial E} = \frac{m}{\pi}\int_{r_-}^{r_0} \frac{dr}{\sqrt{\cdots}}$$

$$= \frac{1}{2\pi} \cdot \frac{\sqrt{2m(E+V_0)r_0^2 - \alpha_\theta^2}}{E+V_0}$$

2. $H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + mgy$



: time-independent, cyclic in x
+ separable in x, y.

- so try $S = \alpha_x x + W_1(y) - Et$

$$\rightarrow \frac{1}{2m}\left(\alpha_x^2 + \left(\frac{\partial W_1}{\partial y}\right)^2\right) + mgy_1 = E.$$

$$W_1 = \int^y dy' \sqrt{2mE - \alpha_x^2 - 2m^2 g y}$$

$$= \frac{-1}{3m^2 g}\left(2mE - \alpha_x^2 - 2m^2 g y\right)^{3/2}.$$

The new (constant) coordinates are $Q_i = \frac{\partial S}{\partial \alpha_i} = \beta_i$

with $\{\alpha_i\} = \{E, \alpha_x\}$ so

$$\beta_E = \frac{\partial}{\partial E}\left(\alpha_x x + W_1 - Et\right) = \frac{-1}{mg}\sqrt{2mE \cdots} - t$$

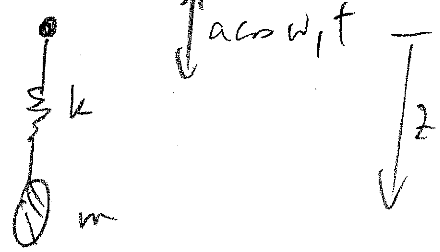
$$\beta_x = \frac{\partial}{\partial \alpha_x}\left(\quad '' \quad \right) = x + \frac{\alpha_x}{m^2 g}\sqrt{2mE \cdots}$$

The old momenta are $p_i = \frac{\partial S}{\partial q_i}$

or $p_x = \frac{\partial S}{\partial x} = \alpha_x$, $p_y = \frac{\partial S}{\partial y} = \sqrt{2mE - \alpha_x^2 - 2m^2 g y}$

Then determine the constants α_i, β_i from the initial conditions on x, y, p_x, p_y ...

3.



$$H = p^2/2m + \tfrac{1}{2}k\left(z - a\cos\omega_1 t\right)^2$$

$$= \frac{p^2}{2m} + \tfrac{1}{2}kz^2 - kza\cos\omega_1 t + \cancel{\tfrac{1}{2}ka^2\cos^2\omega_1 t}$$

$$= H_0 - H_1$$

for H_0, HJ solution is $z = \sqrt{\frac{2\alpha}{k}}\sin\omega_0(t+\beta)$

where $\alpha \leftrightarrow$ energy, $\omega_0 = \sqrt{k/m}$

so $H_1 = -ka\cos\omega_1 t \cdot \sqrt{\frac{2\alpha}{k}}\sin\omega_0(t+\beta)$

$$\dot{\alpha}_1 = -\frac{\partial H_1}{\partial\beta}\Big|_0 = -ka\omega_0\sqrt{\frac{2\alpha_0}{k}}\cos\omega_1 t\cos\omega_0(t+\beta_0)$$

$$\dot{\beta}_1 = \frac{\partial H_1}{\partial\alpha}\Big|_0 = -\frac{ka}{\sqrt{2\alpha_0 k}}\cos\omega_1 t\sin\omega_0(t+\beta_0)$$

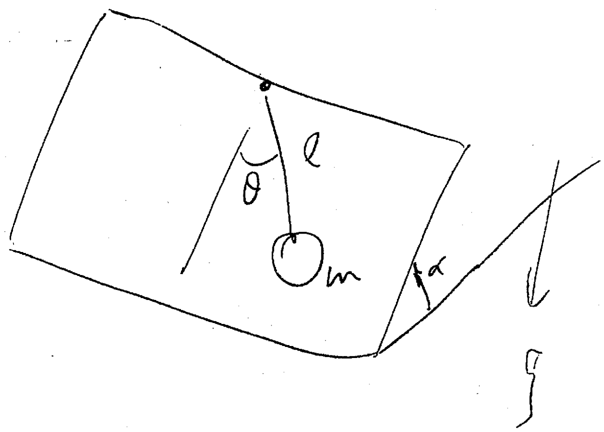
$$\rightarrow \dot{\alpha}_1 = -ka\sqrt{\frac{\alpha_0}{2m}}\Big\{\cos\left[(\omega_0+\omega_1)t + \omega_0\beta_0\right] - \cos\left[(\omega_0-\omega_1)t + \omega_0\beta_0\right]\Big\}$$

$$\alpha_1 = ka\sqrt{\frac{\alpha_0}{2m}}\Big\{\frac{1}{\omega_0+\omega_1}\sin\left[(\omega_0+\omega_1)t + \omega_0\beta_0\right] - \frac{1}{\omega_0-\omega_1}\sin\left[(\omega_0-\omega_1)t + \omega_0\beta_0\right]\Big\}$$

& similarly for β_1.

Then $z(t) = \sqrt{\frac{2}{k}(\alpha_0+\alpha_1)}\sin\omega_0(t+\beta_0+\beta_1)$

4.



θ in plane

α from horizontal

$$V = mgl(1-\cos\theta)\cdot\sin\alpha$$

$$\cong \frac{1}{2} mgl\sin\alpha\cdot\theta^2$$

$$H = \frac{p^2}{2ml^2} + \frac{1}{2} mgl\sin\alpha\cdot\theta^2 = J\omega = J\sqrt{\frac{g\sin\alpha}{l}}$$

Maximum amplitude θ_{max} when $p = 0$ or

$$\theta_{max} = \sqrt{\frac{2J\omega}{mgl\sin\alpha}} = f(J)\cdot(\sin\alpha)^{-\frac{1}{4}}$$

If α varies slowly, $J \cong$ constant, $\theta_{max} \sim (\sin\alpha)^{-\frac{1}{4}}$

or $$\frac{\delta\theta_{max}}{\theta_{max}} = -\frac{1}{4}\cot\alpha\cdot\delta\alpha$$