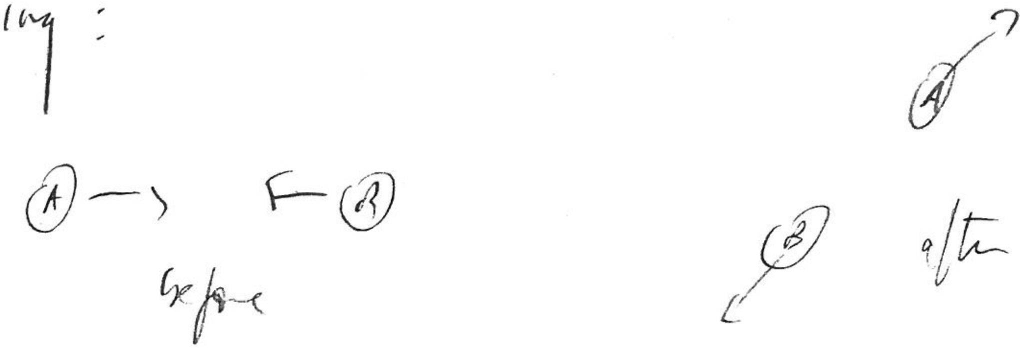
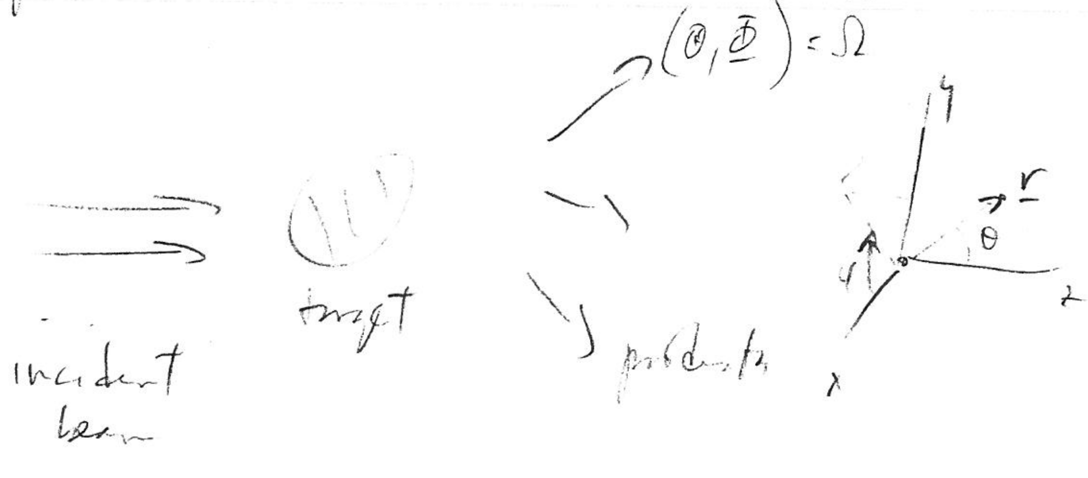
Scattering:



3 aspects to the problem:

1. relate "scattering probability" to classical orbits; note while each orbit is deterministic the initial positions are usually only specified by probabilities.
2. relate orbit to potential ("done already")
3. Transform from relative coord where orbit calc is done to lab reference frame.

Typical Expt:



w/o the beam (e.g. astrophysics) think of individual orbits

Here think of cloud of incident particles; typically target (1 particle at a time) is small compared to beam width so assume <u>uniform</u> incident beam.

$F \equiv$ incident flux $=$ # incoming particles / time / cross-sectional area $\perp$ to beam

$$[F] = \frac{1}{L^2 T}$$

Scattering changes direction (also energy - see below, + ~~particle #~~ + type - see QFT), so let

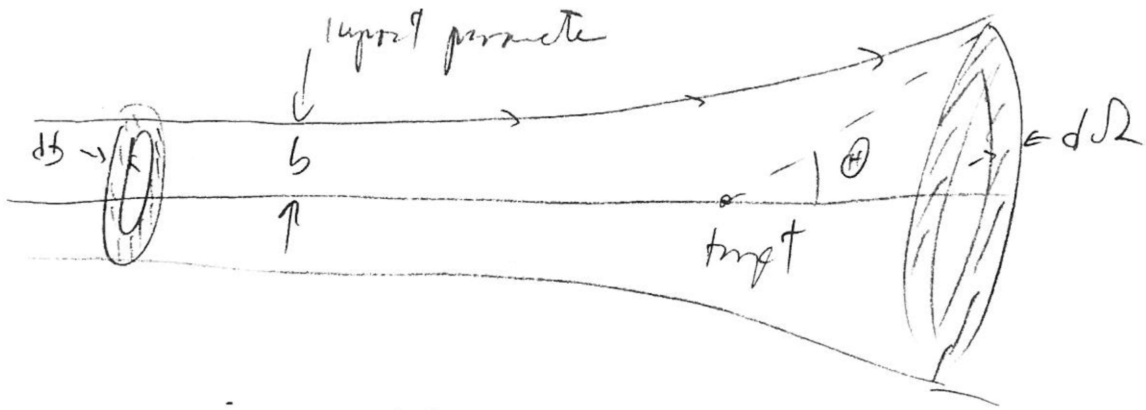
$$\sigma(\hat{\Omega})\, d\Omega \to \sigma(\Theta, \Phi) \sin\Theta \, d\Theta \, d\Phi$$

= # particles scattered into solid angle element $d\Omega$ at $\hat{\Omega}$, per time, per incident flux

$[\sigma] =$ area $\to$ differential cross-section

+ $\int d\Omega \, \sigma(\hat{\Omega}) = \sigma_{TOT}$ = total prob of scatt per flux per time

Central forces: indep of rotation about incident axis ($\hat{z}$) so $\sigma = \sigma(\theta)$



for a given $U(r)$ $\quad b \leftrightarrow \Theta$

so pdf in b $\leftrightarrow$ pdf in Θ

incident in ring of thickness db

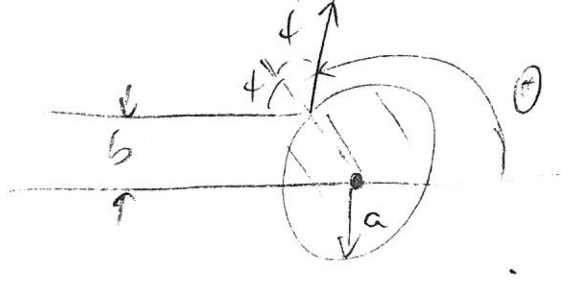
= # scattered in ring at Θ of "thickness" $d\Omega$

per time: $\quad 2\pi b\, db\, F = 2\pi \sin\Theta\, d\Theta \cdot \sigma(\Theta) \cdot F$

$$\rightarrow \quad \sigma(\Theta) = \frac{b}{\sin\Theta} \left| \frac{db}{d\Theta} \right|$$

(need $|\cdot|$ because $\Theta \uparrow \Leftrightarrow b \downarrow$, & $\sigma > 0$)

Example: point particle scattered elastically off a smooth solid sphere



$$\sin\psi = \frac{b}{a}$$

If the sphere is frictionless it can only exert a normal force on the particle so $\vec{p} \cdot \hat{t} = \vec{p}\,' \cdot \hat{t}$ & if the collision is elastic $\psi' = \psi$.

so $\Theta = \begin{cases} \pi - 2\psi & b < a \\ 0 & b > a \end{cases}$

$$b = a\sin\psi = a\sin\frac{\pi - \Theta}{2} = a\cos\frac{\Theta}{2}$$

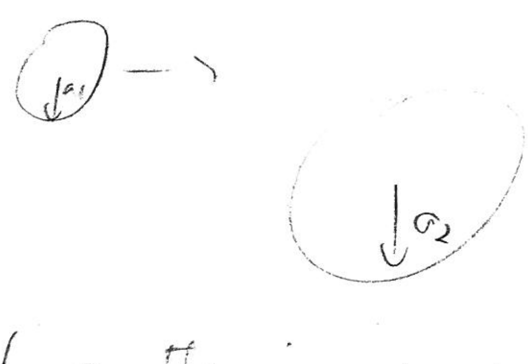
$$\sigma(\Theta) = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| = \frac{a\cos\frac{\Theta}{2}}{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\left|-\frac{a}{2}\sin\frac{\Theta}{2}\right|$$

$$= \frac{a^2}{4}$$

$$\sigma_{tot} = \int \sigma(\Theta)\, d\Theta \cdot \sin\Theta\, d\Phi = \pi a^2$$

: area of the incident beam that is "removed" by the obstacle.

1a. NB: with 2 rigid frictionless elastic spheres



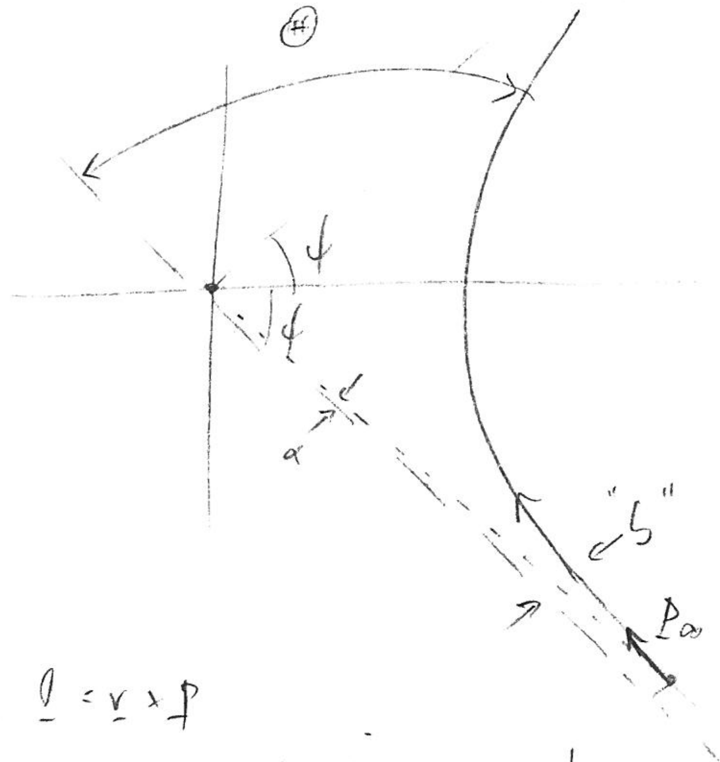
$$\sigma(\Theta) = \frac{\pi}{4}(a_1 + a_2)^2$$

2. Rutherford scattering: $U(r) = \frac{qq'}{4\pi\epsilon_0 r}$ MKS units

$$qq' > 0 \Rightarrow \frac{1}{r} = \frac{\mu k}{\ell^2}(-1 + e\cos\theta)$$

where $k = \frac{qq'}{4\pi\epsilon_0}$ $\quad e = \sqrt{1 + 2E\ell^2/\mu k^2}$

note $e > 1$ so asymptotes $\pm\psi = \pm\cos^{-1}\left(\frac{1}{e}\right) \in \Theta_{1,4}$



$l = \mu v_0 \cdot b$

$v_0 =$ vel as $r \to \infty$
$= \sqrt{2mE}$

$b =$ impact param.
(as $r \to \infty$)

or, $\underline{l} = \underline{r} \times \underline{p}$

$\xrightarrow[r\to\infty]{} r \mu v_0 \sin\alpha = \mu v_0 b.$

$$\text{so} \quad b = \frac{l}{\mu v_0} = \frac{1}{\mu v_0}\sqrt{\frac{\mu k^2}{2E}(e^2-1)}$$

$$= \frac{k}{\mu v_0}\sqrt{\frac{\mu}{2E}}\;\underbrace{\sqrt{\frac{1}{\cos^2\psi} - 1}}$$

$$\left|\frac{\sin\psi}{\cos\psi}\right| = \left|\frac{\sin\frac{\pi-\Theta}{2}}{\cos\frac{\pi-\Theta}{2}}\right| = \underbrace{\left|\cot\frac{\Theta}{2}\right|}$$

drop $|\cdot|$ for $\Theta < \pi$

$$\& \quad \left|\frac{db}{d\Theta}\right| = \frac{k}{\mu v_0}\sqrt{\frac{m}{2E}}\;\frac{1}{2\sin^2\frac{\Theta}{2}}$$

$$\therefore \quad \sigma(\Theta) = \frac{1}{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}} \cdot \frac{k}{\mu v_0}\sqrt{\frac{\mu}{2E}}\left|\cot\frac{\Theta}{2}\right| \cdot \frac{k}{\mu v_0}\sqrt{\frac{\mu}{2E}}\;\frac{1}{2\sin^2\frac{\Theta}{2}}$$

but $E = \frac{1}{2}\mu v_0^2$

so $\sigma(\Theta) = \left(\frac{k}{4E}\right)^2 \frac{1}{\sin^4\frac{\Theta}{2}}$ Rutherford formula

note $\sigma_{TOT} = \infty$! everything scattered, esp at large b / small Θ.

2. General form for the scattering angle:

use conservation of $\begin{cases} E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \\ \ell = \mu r^2 \dot{\theta} \end{cases}$

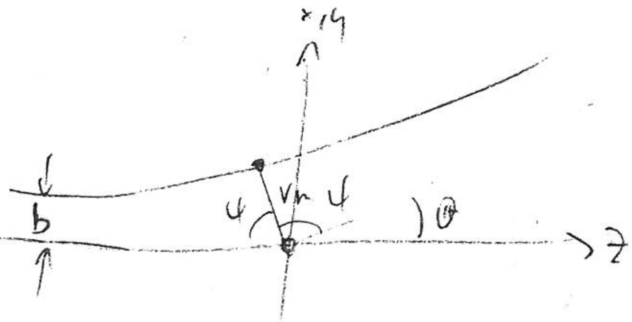
$$\theta = \int d\theta = \int \frac{d\theta}{dt}\frac{1}{dr/dt}\, dr$$: convert to orbit variables

$$= \int \frac{\ell}{\mu r^2} \frac{\pm 1}{\sqrt{\frac{2}{\mu}(E - V(r)) - \ell^2/\mu^2 r^2}}\, dr$$

~> note here for next page

then use $\ell = \mu v_0 b = \sqrt{2\mu E}\, b$

$$\theta = \theta_0 \pm b \int_{r_0}^{r(\theta)} \frac{dr/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}.$$



Choose $\theta_0 = \pi$, $r_0 = \infty$, − sign

so $\theta = \pi - b\int_{r(\theta)}^{\infty} \dots$

Notice at $r = r_m$ = min radius

$\theta = \pi - \psi$

so $\quad \pi - \psi = \pi - b \int_{r_m}^{\infty} \cdots$

$$\text{+} \quad \Theta = \pi - 2\psi = \pi - 2b \int_{r_m}^{\infty} \frac{dr/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}$$

Note: tacitly assumed that orbit is symmetric about r_m

proof: back to $\theta = \theta_0 \pm \frac{\ell}{\mu} \int_{r_0}^{r(\theta)} \frac{dr}{r^2} \cdots$

at r_m, $\dot{r} = 0$ so $V_{eff}(r) = V(r) - \frac{\ell^2}{\mu^2 r^2} \xrightarrow[r \to r_m]{} E$

choose $r_0 = r_m$, $\theta_0 = \psi$:

$$\theta - \psi = \pm \int_{r_m}^{r} \frac{dr/r^2}{\sqrt{2\mu\left(V_{eff}(r_m) - V_{eff}(r)\right)}}$$

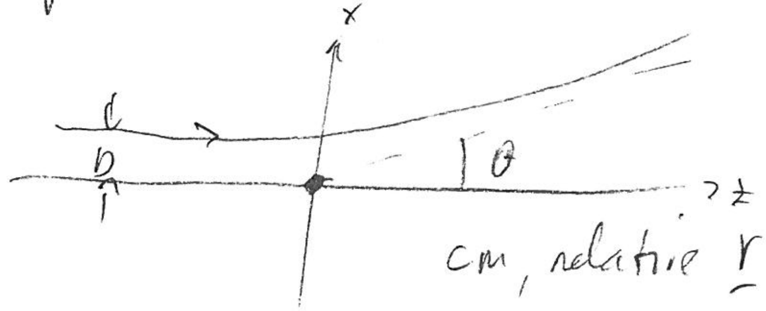
$\pm$ for $\theta \gtrless \psi$

so same value of r gives same $|\theta - \psi|$:

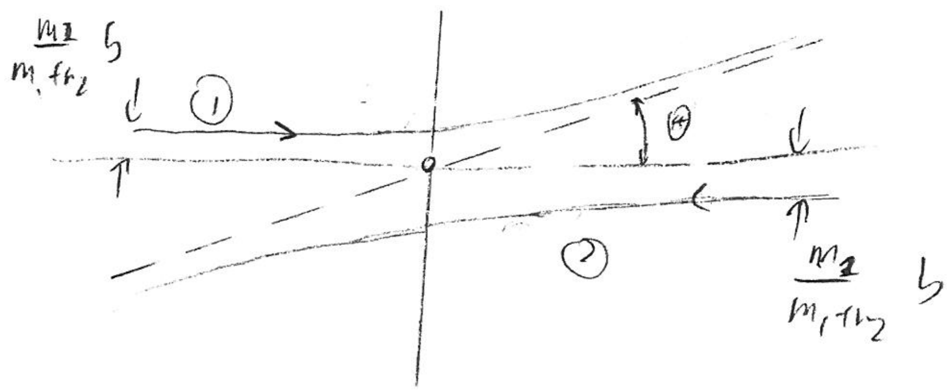
orbit symmetric about (r_m, ψ)

Coordinate transformation:

just did

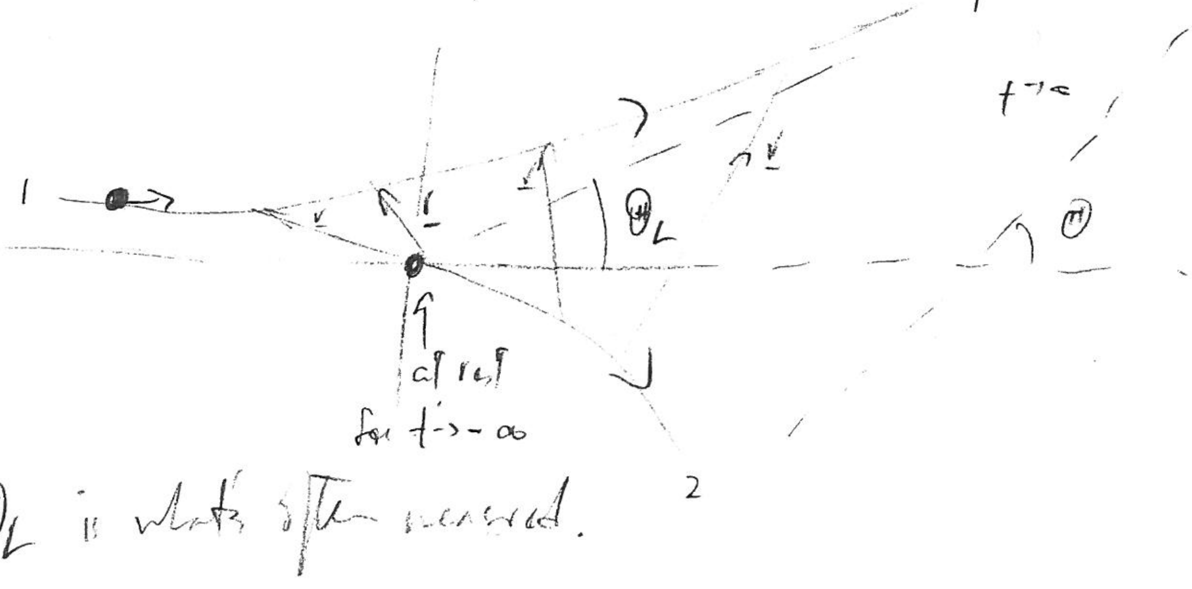


in terms of $\underline{r}'_{1,2}$ in cm, $\underline{r}'_1 = \frac{m_2}{m_1+m_2}\underline{r}$, $\underline{r}'_2 = \frac{-m_1}{m_1+m_2}\underline{r}$



Same Θ: angle btw initial & final relative velocities

Want to go to lab system (2 at rest initially):



Θ_L is what's often measured.

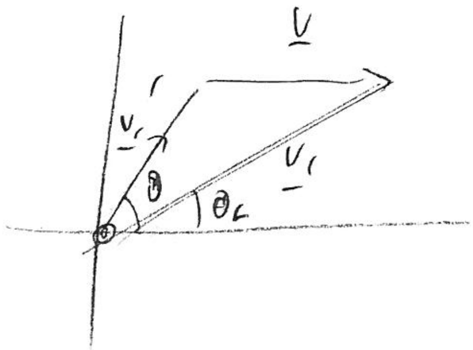
let $\underline{r}_i, \underline{v}_i$ = final pos/vel of i in lab system

$\underline{r}_i', \underline{v}_i'$ = " " " CM

$\underline{R}, \underline{V}$ = initial a final pos/vel of CM in lab.

Use conservation of p:

in lab: $m_1 \underline{v}_0 = (m_1 + m_2)\underline{V} \to \underline{V} = \frac{\mu}{m_2}\underline{v}_0$



$$\underline{v}_1 = \underline{v}_1' + \underline{V}$$

$$v_1 \sin\Theta_L = v_1' \sin\Theta$$

$$v_1 \cos\Theta_L = v_1' \cos\Theta + V$$

so $\tan\Theta_L = \dfrac{\sin\Theta}{\cos\Theta + \rho}$

relative speed $|\dot{r}|$

where $\rho = \dfrac{V}{v_1'} = \dfrac{\mu}{m_2} v_0 \cdot \left(\dfrac{\mu v}{m_1}\right)^{-1} = \dfrac{m_1}{m_2}\dfrac{v_0}{v}$

Note if $m_1 = m_2$ + elastic so $v = v$ $\to$ $\tan\Theta_L = \tan\frac{\Theta}{2}$

so $\Theta_L = \frac{\Theta}{2} \le \pi/2$

— x —

What if energy was not conserved, e.g.

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\mu v_0^2 + Q$$

final cm energy, initial cm energy

$$Q = \begin{cases} \text{pos} & \text{- explosion ?} \\ \text{neg} & \text{- adsorption/deformation ?} \end{cases}$$

then $\dfrac{v^2}{v_0^2} = 1 + \dfrac{2Q}{\mu v_0^2} = 1 + \dfrac{m_1}{\mu}\dfrac{Q}{E_L}$

$E_L = \frac{1}{2} m_1 v_0^2$ = initial lab energy

$$\rho = \frac{m_1}{m_2}\frac{v_0}{v} = \frac{m_1}{m_2\sqrt{1 + m_1 Q/\mu E_L}}$$

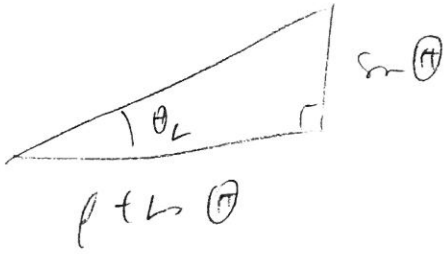
Finally: transform σ to lab frame:

particles/time scattered into solid $\angle$ element about Θ in com frame

= # scattered into transformed element about θ_L in lab frame

$$\left| \sigma(\Theta) \cdot I \cdot 2\pi \sin\Theta \, d\Theta \right|$$
$$= \left| \sigma_L(\Theta_L) \cdot I \cdot 2\pi \sin\Theta_L \, d\Theta_L \right|$$

or $\sigma_L(\Theta_L) = \sigma(\Theta) \left| \dfrac{d(\cos\Theta)}{d(\cos\Theta_L)} \right|$



$$\cos\Theta_L = \frac{\rho + \cos\Theta}{\sqrt{1 + 2\rho\cos\Theta + \rho^2}}$$

$$\vdots$$

$$\sigma_L(\Theta_L) = \sigma(\Theta) \cdot \frac{\left(1 + 2\rho\cos\Theta + \rho^2\right)^{3/2}}{1 + \rho\cos\Theta}$$

Also must have $\sigma_{tot} = \sigma_{tot,L}$