Oscillations – one degree of freedom { spring, pendulum, etc.

"etc" means look at behavior of a forced system near equilibrium

$$V(x) = V(x_0) + \underbrace{V'(x_0)}_{=0}(x-x_0) + \tfrac{1}{2}\underbrace{V''(x_0)}_{=k\ \text{stable eq}}(x-x_0)^2 + \underbrace{O(x-x_0)^3}_{\text{small}}$$

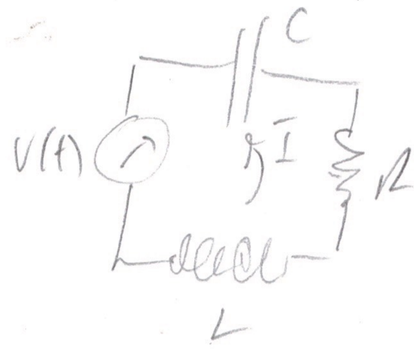
x = whatever the variable is

add damping: environment resists motion $F = -\lambda\dot{x}$

add external force $F(t)$ [$F(x,t)$ has no general theory]

$$\rightarrow m\ddot{x} = -kx - \lambda\dot{x} + F(t)$$

also works for electrical system



$$L\dot{I} + RI + \tfrac{1}{C}Q = V(t)$$

$$\xrightarrow{I=\dot{Q}} L\ddot{Q} + R\dot{Q} + \tfrac{1}{C}Q = V(t)$$

same eq. as above.

Special cases:

1. no damping no force $\lambda = F = 0$ "SHO"

e.g. $x(t) = x_0\cos\omega_0 t + \dot{x}_0/\omega_0 \sin\omega_0 t \qquad \omega_0 = \sqrt{\tfrac{k}{m}}$

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \text{constant}$$

Oscillates forever, no loss of energy

2. damping + no force at all

$$m\ddot{x} + \lambda\dot{x} = 0 \qquad E = \tfrac{1}{2}m\dot{x}^2 \left(+\tfrac{1}{2}kx^2\right)$$

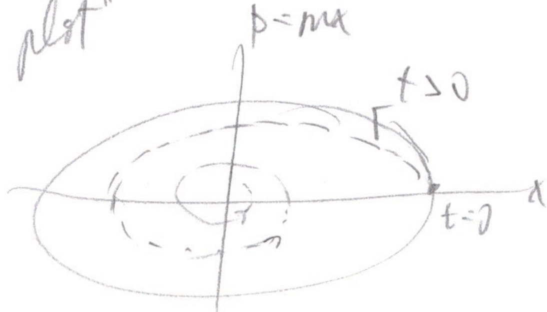
$$x(t) = A + B e^{-\lambda t/m} \xrightarrow[t\to\infty]{} \text{constant}$$

consistent

$$\frac{dE}{dt} = \frac{d}{dt}\left[\tfrac{1}{2}m\dot{x}^2 \left(+\tfrac{1}{2}kx^2\right)\right]$$

$$= \dot{x}\left[m\ddot{x} \left(+kx\right)\right] = -\lambda\dot{x}^2 \quad \text{so } E \xrightarrow[t\to\infty]{} 0$$

"phase plane plot"



— $\lambda = 0$

--- $\lambda \neq 0$

———— x ————

Rewrite as $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ for $f = 0$

$$\beta = \lambda/2m \qquad \omega_0^2 = k/m$$

look for solutions $x = Ae^{i\omega t}$

$$\left(-\omega^2 + 2i\beta\omega + \omega_0^2\right) Ae^{+i\omega t} = 0$$

$$\to \omega_\pm = i\beta \pm \sqrt{\omega_0^2 - \beta^2}$$

so $x(t) = A_+ e^{i\omega_+ t} + A_- e^{i\omega_- t}$ $\quad A_\pm \leftrightarrow$ initial data

Note $x(t) \propto e^{-\beta t}$ so damping $\leftrightarrow$ decay of motion

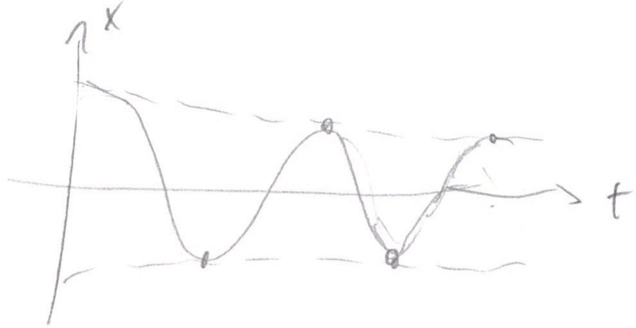
Limits: $\beta << \omega_0$ - weak damping

$$\sqrt{\beta^2-\omega_0^2} = i\sqrt{\omega_0^2-\beta^2} = i\omega_0\left[1-\beta^2/2\omega_0^2+\cdots\right]$$

$$\equiv i\omega_1$$

$$x(t) = e^{-\beta t}\left[A_+ e^{i\omega_1 t} + A_- e^{-i\omega_1 t}\right]$$

"underdamped" $\longrightarrow A_0 \underbrace{e^{-\beta t}}_{\text{weak decay}} \underbrace{\cos(\omega_1 t + \delta)}_{\text{oscillates at shifted freq}}$



$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots$$

or, if $\beta >> \omega_0$ - strong damping

$$\sqrt{\beta^2-\omega_0^2} \equiv \omega_2 = \beta\left(1-\omega_0^2/2\beta^2+\cdots\right)$$

notice $\omega_2 < \beta$ so $-\beta \pm \omega_2 < 0$

$$x(t) = e^{-\beta t}\left[A_+ e^{-\omega_2 t} + A_- e^{+\omega_2 t}\right]$$

= pure decay (recover sp. case for $\omega_0 \to 0$)

"overdamped"

if $\beta = \omega_0$ "critical damping"

2 exp solutions degenerate

Th of ODE → $x(t) = Ae^{-\beta t} + Bte^{-\beta t}$

still decays eventually but could go ±.

Forced oscillations:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \qquad f = F/m$$

Th of ODEs says:

$x(t) = x_h(t) + x_p(t)$

↳ particular solution of forced eq

↳ general solution of homogeneous eq.

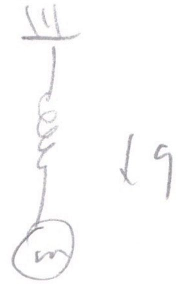
Simple case $f = \text{const} = f_0$

obvious $x_p(t) = f_0/\omega_0^2$

since $x_h(t) \to 0$ as $t \to \infty$, $x(t) \to \frac{f_0}{\omega_0^2}$ as $t \to \infty$

& the equil pos shifts

e.g.



x_0 = length at $g = 0$

$k(x - x_0) = mg$ with grav

↳ $x = x_0 + mg/k$

Now take $f(t) = f_0 \cos\omega t \to f_0 e^{i\omega t}$

(1) oscillatory forcing is common

(2) any $f(t) = \sum_n f_n e^{2\pi i n t/T}$ for $0 \le t \le T$

as a Fourier series

$x(t)$ eq is linear: $x(t) = \sum_n x_n(t)$

where x_n = solution for $f_n e^{\cdots}$

So $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 e^{i\omega t}$

as before $x(t) = x_h(t) + x_p(t)$

Expect to see the system oscillating at the same freq.

as the force, since $x_h \to 0$ as $t \to \infty$

so look for $x_p(t) = C e^{i\omega t}$

$$\to \quad C(-\omega^2 + 2i\beta\omega + \omega_0^2) = f_0$$

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = f_0 \frac{\omega^2 - \omega_0^2 - 2i\beta\omega}{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}$$

C is big when $\omega \simeq \omega_0$ = "natural freq"

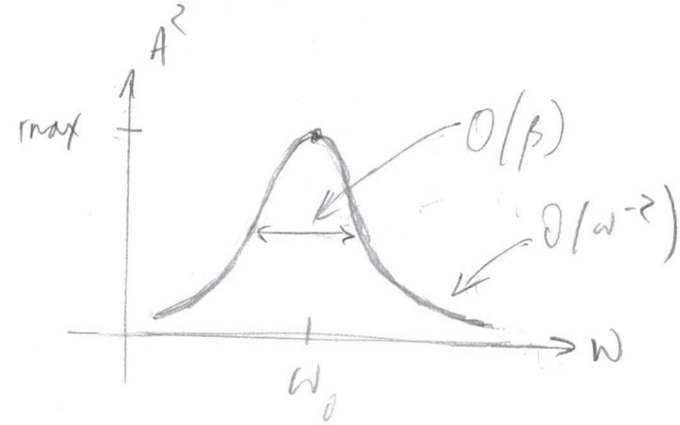
Useful to write $C = A e^{-i\delta}$ $\quad A$ = amplitude

δ = phase lag

$$A^2 = C^*C = \frac{f_0^2}{(\omega^2-\omega_0^2)^2+4\beta^2\omega^2}, \quad \tan\delta = \frac{2\beta\omega}{\omega_0^2-\omega^2}$$

$$\Rightarrow \quad x(t) = \frac{f_0\cos(\omega t-\delta)}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}} + \text{exp. decaying terms.}$$

↑ lose memory of IC



resonant behavior

biggest response when appl. freq = natural freq.

width ↔ how selective the response is

note $A^2|_{max} = \dfrac{f_0^2}{4\beta^2\omega^2}$

+ when $\omega = \omega_0 \pm \beta$

$$(\omega_0^2-\omega^2)^2 = (\omega_0+\omega)^2(\omega_0-\omega)^2 = (2\omega_0\pm\beta)^2\beta^2$$

$$\simeq 4\beta^2\omega_0^2 \quad \text{if } \omega_0 \gg \beta$$

$$\rightarrow \quad A^2(\omega_0\pm\beta) = \tfrac{1}{2}A^2|_{max}$$

So $2\beta \equiv$ "Full width at half-maximum"

Common characterization of oscillators:

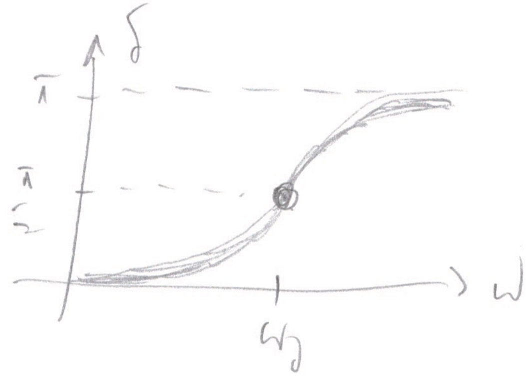
def "quality factor" $Q = \dfrac{\omega_0}{2\beta} = \dfrac{\text{natural freq}}{\text{FWHM}}$

$$= \pi \cdot \frac{1/\beta}{2\pi/\omega_0} = \pi \cdot \frac{\text{decay time}}{\text{osc period}}$$

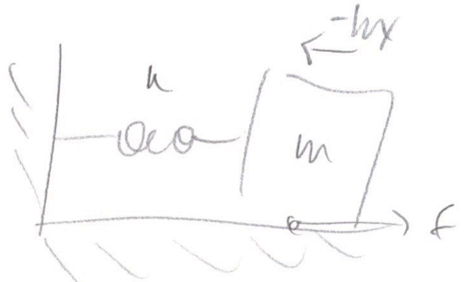
large Q ⟺ narrow resonance ⟺ selective response

Behavior of phase: $\tan\delta = \dfrac{2\beta\omega}{\omega_0^2 - \omega^2}$

since



What about solid-on-solid friction?



take $x = 0$ for unstretched length

start from rest at $x_0 > 0$

assume $kx_0 > \mu_s mg > \mu_k mg$

$$m\ddot{x} = +\mu_k mg - kx$$

or $$m \frac{d^2}{dt^2}\left(x - \mu_k mg/k\right) = -k\left(x - \underbrace{\mu_k mg/k}_{\equiv \xi}\right)$$

$$\to \quad x(t) = \xi + (x_0 - \xi)\cos\omega_0 t$$

$$\dot{x}(t) = -\omega_0 (x_0 - \xi)\sin\omega_0 t$$

Mass stops at $t^* = \frac{\pi}{\omega_0}$ where $x(t^*) = 2\xi - x_0 > -x_0$

(doesn't get all the way to $-x_0$ due to frictional energy loss)

If $|kx(t^*)| < \mu_s mg$: stops

else : repeat calc with smaller initial displacement

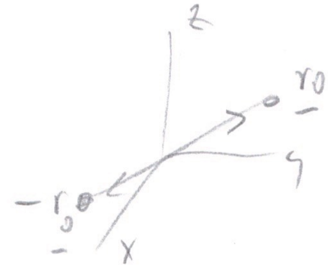
$\to$ finite # of decaying osc until it stops at $x \neq 0$.

Higher dimensions:

simple case $\underline{F} = -k\underline{r}$ (isotropic spring)

$x(t) = x_0 \cos(\omega t + \delta_x)$ if at rest at $t=0$ ($\underline{\delta}=0$)

etc. $\rightarrow$ motion along $\underline{r}_0$



if $\dot{\underline{r}}(0)$ not along $\underline{r_0}$ $\rightarrow$ elliptical motion

e.g. $x = x_0 \cos \omega t$, $y = y_0 \cos(\omega t + \delta)$ $\rightarrow$
- line if $\delta = 0$
- ellipse if $\delta = \pi/2$
- tilted ellipse otherwise

Anisotropic case: $F_x = -k_x x$, $F_y = -k_y y$ $\rightarrow$ $x = x_0 \cos(\omega_x t + \delta_x)$, $y = y_0 \cos(\omega_y t + \delta_y)$

if $\frac{\omega_x}{\omega_y} = \frac{n}{m}$ i.e. $\frac{k_x}{k_y} = (\text{rational } \#)^2$ then

periodic motion: returns to start point

proof: let $\omega_x = n\Omega$, $\omega_y = n\Omega$ so

at $t = \frac{2\pi}{\Omega}$ $\omega_x t = 2\pi n$ & $\omega_y = 2\pi m$

otherwise not

Effects of non-linearity:

Suppose $\quad m\ddot{x} + \cancel{\gamma\dot{x}} + kx = \varepsilon x^3$

drop $\qquad\qquad$ small

(turns out x^3 simpler / more instructive than x^2)

$\rightarrow \quad \ddot{x} + \omega_0^2 x = \lambda x^3 \qquad \omega_0^2 = \frac{k}{m} \qquad \lambda = \frac{\varepsilon}{m}$ small

Try $\quad x(t) = A\cos\omega t \quad$ with new freq ω!

$$-\omega^2 A\cos\omega t + \omega_0^2 A\cos\omega t = \frac{\lambda A^3}{4}\left[\cos 3\omega t + 3\cos\omega t\right]$$

trig identity.

or

$$\left(\omega_0^2 - \omega^2 - \frac{3\lambda A^2}{4}\right) A\cos\omega t = \frac{\lambda A^3}{4}\cos 3\omega t$$

which suggests that a small $\cos 3\omega t$ term is needed.

Better: $\quad x(t) = A\cos\omega t + B\cos 3\omega t \qquad |B| \ll A$

$$\left(\omega_0^2 - \omega^2\right) A\cos\omega t + \left(\omega_0^2 - 9\omega^2\right) B\cos 3\omega t$$

$$= \frac{\lambda A^3}{4}\left[\cos 3\omega t + 3\cos\omega t\right] + \frac{3\lambda A^2 B}{4}\cos\omega t\cos 3\omega t + \cdots$$

or

$$\left(\omega_0^2 - \omega^2 - \frac{3}{4}\lambda A^2\right) A\cos\omega t + \left[B\left(\omega_0^2 - 9\omega^2\right) - \frac{1}{4}\lambda A^3\right]\cos 3\omega t$$

$$= \text{terms in } \lambda B,\ \lambda B^2 \cdots = \text{smaller} \ : \ 2^{nd} \text{ order}$$

So $\omega^2 = \omega_0^2 - \frac{3}{4}\lambda A^2$

$\omega = \omega_0 - 3\lambda A^2/8\omega_0 + \cdots$

and $B = \dfrac{\frac{1}{4}\lambda A^3}{\omega_0^2 - 9\omega^2} = \dfrac{\frac{1}{4}\lambda A^3}{\frac{27}{4}\lambda A^2 - 9\omega_0^2 + \omega_0^2} \simeq -\dfrac{\lambda A^3}{32\omega_0^2}$

$\rightarrow \quad x(t) = A\cos\omega t - \dfrac{\lambda A^3}{32\omega_0^2}\cos 3\omega t$

Note: $\left.\begin{array}{l}\text{period depends on } A \\ \text{motion } \underline{\text{not}} \text{ simple harmonic}\end{array}\right\}$ differs from SHO