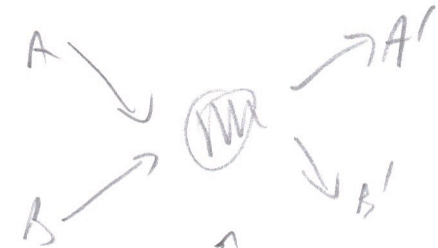
Scattering $A + B \to C + D$ (+E+F+... sometimes)

common experimental method

"look at" the final state & try to deduce target structure or nature of interactions or ...

Here: focus on $2 \to 2$ elastic case $A + B \to A' + B'$



same particles before & after but new momenta & KE

Assume "short-range" interactions which act in finite region, free motion outside

True for contact forces, nuclear forces, ..

(Not true for $V \sim \frac{1}{r}$ - Coulomb, gravity ... - get orbits)

1-d case:

$$\left.\begin{aligned} m_A v_A + m_B v_B &= m_A v_A' + m_B v_B' \\ \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2 &= \tfrac{1}{2} m_A v_A'^2 + \tfrac{1}{2} m_B v_B'^2 \end{aligned}\right\} \begin{aligned} &\text{cons of } p \\ &\text{cons of } E \end{aligned}$$

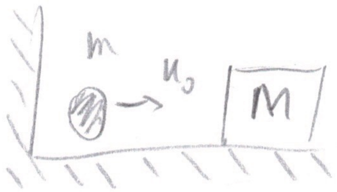
Could solve for v_A' & v_B' given v_A & v_B ...

$\to$ rewrite
$$\left\{\begin{aligned} m_A (v_A - v_A') &= m_B (v_B' - v_B) \\ m_A (v_A^2 - v_A'^2) &= m_B (v_B'^2 - v_B^2) \end{aligned}\right.$$

divide: $v_A + v_A' = v_B + v_B'$

or $(v_A - v_B) = -(v_A' - v_B')$: relative vel. reverses

1-d example:



Ball of mass m + velocity u_0 collides elastically with block of mass $M \gg m$ elastically, bounces back + forth btw. block + wall.

Friction acts on M, large enough to bring it to rest after each collision before the next one; no friction on m.

How far does M get + how long does it take?

let u_n = ball velocity after n^{th} collision
to left just after ($m \ll M$)
to right before next coll (vel. reverses on wall)

V_n = block vel after n^{th} collision

Then $\quad m u_n + 0 = -m u_{n+1} + M V_{n+1}$

block at rest $\quad$ ball recoils after collision

Also $\quad 0 - u_n = -(V_{n+1} + u_{n+1})$: rel vel reverses

So $\quad m u_n = -m u_{n+1} + M(u_n - u_{n+1})$

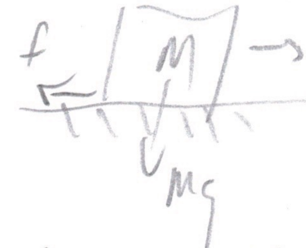
or $\quad u_{n+1} = \dfrac{M-m}{M+m} u_n = \dfrac{1-\varepsilon}{1+\varepsilon} u_n \quad \varepsilon \equiv m/M \ll 1$

$\simeq (1-2\varepsilon) u_n = (1-2\varepsilon)^n u_0 \simeq e^{-2\varepsilon n} u_0$

$$v_{n+1} = u_n - u_{n+1} = u_0 e^{-2\varepsilon n}\left[1-(1-2\varepsilon)\right]$$

$$= 2\varepsilon u_0 e^{-2\varepsilon n}$$

After $(n+1)^{st}$ coll:



$$M\dot{v} = f = -\mu Mg$$

$$v(t) = v_{n+1} - \mu g t$$

stops at $t_{n+1} = v_{n+1}/\mu g$

distance: $0 - v_{n+1}^2 = 2(-\mu g)\, d_{n+1} \rightarrow d_{n+1} = \dfrac{v_{n+1}^2}{2\mu g}$

Total time: $\displaystyle\sum_{n=0}^{\infty} t_{n+1} = \frac{1}{\mu g} 2\varepsilon u_0 \underbrace{\sum_{n=0}^{\infty} e^{-2\varepsilon n}}_{= \frac{1}{1-e^{-2\varepsilon}} \simeq \frac{1}{2\varepsilon}}$

$$= \frac{u_0}{\mu g}$$

Total distance $\displaystyle\sum_{n=0}^{\infty} d_{n+1} = \frac{1}{2\mu g}(2\varepsilon u_0)^2 \underbrace{\sum_{n=0}^{\infty} e^{-4\varepsilon_0 n}}_{= \frac{1}{1-e^{-4\varepsilon}} \simeq \frac{1}{4\varepsilon}}$

$$= \frac{\varepsilon u_0^2}{2\mu g}$$

Conservation of energy:

initial KE $\frac{1}{2} m u_0^2$ dissipated by frictional work

$W = \mu Mg \cdot d \quad \rightarrow \quad d = \varepsilon u_0^2 / 2\mu g$ ✓

2d

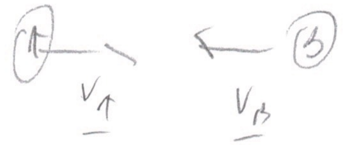
2-body elastic collision $A + B \to A' + B'$

simplify using = masses (general case in Fowles/Marion)

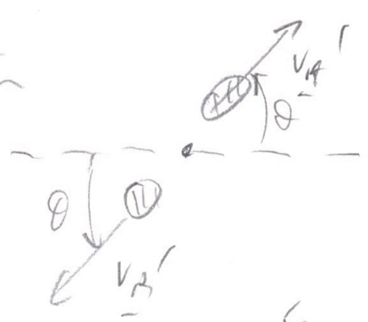
(1) CM frame $\sum_i m_i \underline{v}_i = 0 = \sum_i m_i \underline{v}_i' \to m(\underline{v}_A + \underline{v}_B)$

$= m(\underline{v}_A' + \underline{v}_B')$ by cons. of p

before:



after



$\theta_A = \theta_B$ by cons of p

2 scattering "lines" form a scatt. plane, not nec. in paper

$E_{before} = 2 \times \frac{1}{2} m v_A^2 = E_{after} = 2 \times \frac{1}{2} m v_A'^2$

so $v_A' = v_A = v_B = v_B' = \sqrt{E/m}$

θ = only independent variable (φ irrelevant)

collision probability $\propto$ "scattering cross section" $= f(\theta)$

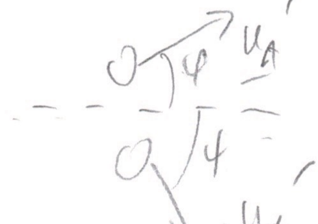
(2) Rest frame of B ("lab frame" for a particle beam)

before:

$E = \frac{1}{2} m u_A^2$

$\underline{p} = m \underline{u}_A$

after



$E = \frac{1}{2} m u_A'^2 + \frac{1}{2} m u_B'^2$

$\underline{p} = m \underline{u}_A' + m \underline{u}_B'$

combine E also $= \frac{1}{2}m(\underline{u}_A' + \underline{u}_B'^2) = \frac{1}{2}m\underline{u}_A'^2 + \underbrace{m\underline{u}_A'\cdot\underline{u}_A'}_{=0} + \frac{1}{2}m\underline{u}_A'^2$

so $\underline{u}_A' \perp \underline{u}_B'$ or $\varphi + \psi = \frac{\pi}{2}$

cons of p $\quad \parallel: \; mu_A = mu_A'\cos\varphi + mu_B'\underbrace{\cos\left(\frac{\pi}{2}-\varphi\right)}_{\sin\varphi}$

$\perp: \; 0 = mu_A'\sin\varphi - mu_B'\underbrace{\sin\left(\frac{\pi}{2}-\varphi\right)}_{\cos\varphi}$

so $u_B' = u_A'\tan\varphi$ & $u_A = u_A'\big(\underbrace{\cos\varphi + \sin\varphi\cdot\tan\varphi}_{1/\cos\varphi}\big)$

or $\left.\begin{aligned} u_A' &= u_A\cos\varphi \\ u_B' &= u_A\sin\varphi \end{aligned}\right\}$ φ is indep variable

$\varphi = 0$: no collision

$\varphi = \pi$: A recoils from B

(3) Relation btw angles?

$\underline{u}_i = \underline{v}_i + \underline{V}_{cm}$ all $i = A, A', B, B'$

apply to A' $\quad \parallel: \; u_A'\cos\varphi = v_A'\cos\theta + V_{cm}$

$\perp: \; u_A'\sin\varphi = v_A'\sin\theta$

apply to B $\quad 0 = -v_B' + V_{cm}$; also $v_B' = v_A'$

divide : $\quad \tan\varphi = \dfrac{\sin\theta}{1+\cos\theta}$

General Solution for 1-d motion

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x) = E = \text{const}$$

so $\dfrac{dx}{dt} = \pm\sqrt{\dfrac{2}{m}\left(E - V(x)\right)}$

$$\rightarrow \quad t - t_0 = \sqrt{\frac{m}{2}}\int_{x(0)}^{x(t)} \frac{dx'}{\sqrt{E - V(x')}}$$

NB: $V(x)$ is undefined up to a const : $V \to V + V_0 \Rightarrow \nabla V \to \nabla V$

E " "

so $E - V(x)$ has no arb. const so no dynamical relevance.

e.g. $V(x) = \frac{1}{2}kx^2$

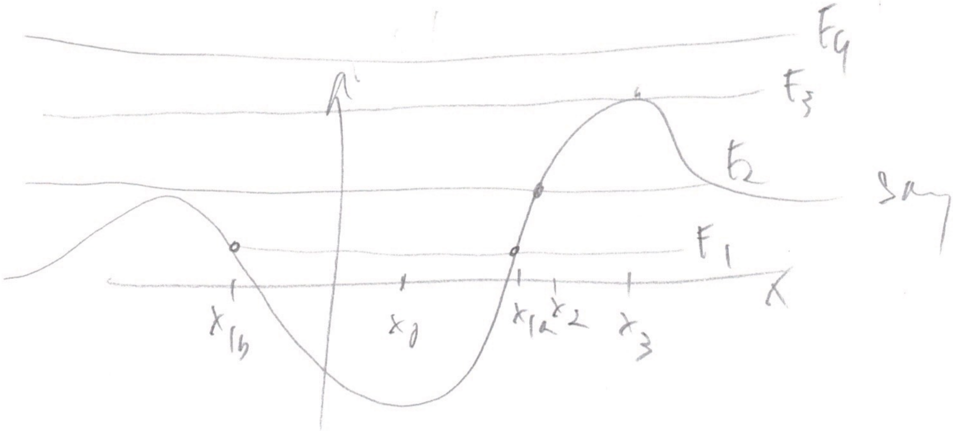
$$t = \sqrt{\frac{m}{2}}\int^x dx' \frac{1}{\sqrt{E - \frac{1}{2}kx'^2}} = \sqrt{\frac{m}{2}}\cdot\sqrt{\frac{2}{k}}\sin^{-1}\sqrt{\frac{k}{2E}}\,x + \text{const}$$

$$\xrightarrow[x(0)=0]{} \quad x(t) = \sqrt{\frac{2E}{m}}\sin\sqrt{\frac{k}{m}}\,t$$

General case: $F(x) \rightarrow V = -\int^x dx'\,F(x') + \text{constant}.$

choose, e.g. $V \to 0$ as $|x| \to \infty$
$V = 0$ at equilibrium pt
anything else

$\Bigg\}$ pure convenience



say the particle starts at x_0 with $\dot{x}_0 > 0$

$$E = E(t=0) = \underbrace{\tfrac{1}{2}m\dot{x}_0^2}_{\text{pos}} + V(x_0) > V(x_0)$$

if at E_4: $\quad \frac{1}{2}m\dot{x}^2 + V(x) > V(x)$ everywhere

$$\dot{x} = + \sqrt{\frac{2}{m}(E - V(x))}$$

starts pos + never hits 0 so always > 0

$\therefore$ escapes to $+\infty$

if $\dot{x}_0 < 0$ escapes to $-\infty$

E_2: region $x > x_2$ has $V > E$ $\quad \dot{x} =$ imaginary : impossible

if $\dot{x}_0 > 0$ $\quad \dot{x} > 0$ for $x < x_2$ then $\dot{x} \to 0$

when x is close to x_2 expand

$$V(x) = \underset{\overset{\parallel}{E_2}}{V(x_2)} + F_2(x - x_2) + O(x - x_2)^2$$

$$F_2 > 0$$

let $x_2 - x = y \Rightarrow \dot{y} = -\sqrt{\frac{2F_1}{m} y}$

$$\sqrt{y(t)} - \sqrt{y_0} = -\sqrt{F_1/2m}\, t$$

so $y \to 0$ at $t^* = \sqrt{2my_0/F_1}$

: particle hits wall in finite time

when there : $E = \frac{1}{2}\dot{x}^2(t^*) + V(x_2)$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{''}{E}$

so $\dot{x}(t^*) = 0$: particle is at rest & feels force F_1 to left

→ particle bounces off x_2 & goes back.

→ goes off to $-\infty$

$E = E_1$: particle bounces back & forth btw x_{1a} & x_{1b}

$E = E_3$: near x_3 $V = E_3$, $V' = 0$ $V'' < 0$

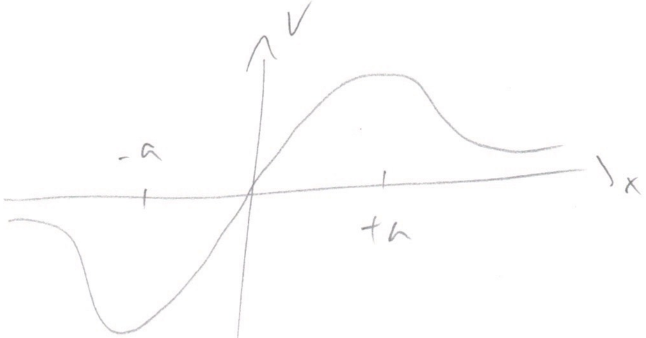
→ $V(x) \cong E_3 - \frac{1}{2}k(x-x_3)^2$, $k > 0$

let $y = x_3 - x$ $\qquad \dot{y} = -\sqrt{k/m}\, y$

$$y = y_0\, e^{-\sqrt{k/m}\, t}$$: $y = 0$ only for $t \to \infty$

: particle slowly climbs the hill & stops at the top

Example: particle moves in $V(x) = \frac{cx}{x^2+a^2}$ $\quad c > 0$



$V' = 0$ at $\frac{c}{x^2+a^2} - \frac{2cx^2}{(x^2+a^2)^2}$

$= \frac{c(a^2-x^2)}{(x^2+a^2)^2} = 0$

So possible equilibria at $x = \pm a$

$V''(x) = -\frac{2cx}{(x^2+a^2)^2} - \frac{4xc(a^2-x^2)}{(\quad)^3} \xrightarrow[x=\pm a]{} \mp \frac{2ca}{4a^2} = \mp \frac{c}{2a}$

So $x = +a$: unstable
$x = -a$ stable

If particle starts at rest at $x = -a$: $E = \frac{1}{2}mv^2 - \frac{c}{2a}$

$\therefore \quad E \begin{cases} < 0, & v < \sqrt{c/m} \text{ : trapped} \\ > V(+a) = \frac{c}{2a} & v > \sqrt{c/m} \text{ : escapes to } +\infty \end{cases}$

$0 < E < \frac{c}{2a}$ & $v < 0$: escapes to $-\infty$.

Oscillation about eqm : $x = -a + \xi \quad |\xi| \ll a$

$m\ddot{\xi} = -V'(-a+\xi) \simeq -V''(-a)\,\xi = -\frac{c}{2a}\xi$

: SHO with frequency $\omega = \sqrt{\frac{c}{2am}}$.