N-particle decomposition:

let $\underline{R}_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i \underline{r}_i$, $\underline{r}_i' \equiv \underline{r}_i - \underline{R}_{cm}$ relative coordinate

$\underline{v}_i' = \underline{v}_i - \underline{V}_{cm}$

notice $\sum_i m_i \underline{r}_i' = 0 = \sum m_i \underline{v}_i'$

$\underline{P}$ relative to CM

$$\underline{P} = \sum_i m_i \dot{\underline{r}}_i = \sum_i m_i \left(\underline{V}_{cm} + \underline{v}_i'\right) = M\underline{V}_{cm} + \underline{P}'$$

$$\underline{L} = \sum_i \underline{r}_i \times m_i \underline{v}_i = \sum_i \left(\underline{R}_{cm} + \underline{r}_i'\right) \times m_i \left(\underline{V}_{cm} + \underline{v}_i'\right)$$

$$= \underline{R}_{cm} \times M\underline{V}_{cm} + \underline{R}_{cm} \times \sum m_i \underline{v}_i' + \sum m_i \underline{r}_i' \times \underline{V}_{cm} + \sum \underline{r}_i' \times m_i \underline{v}_i'$$

$$= \underline{L}_{cm} + \underline{L}'$$ ↖ $\underline{L}$ relative to CM

$$T = \sum_i \frac{1}{2} m_i \underline{v}_i^2 = \sum \frac{1}{2} m_i \left(\underline{V}_{cm} + \underline{v}_i'\right)^2$$

$$= \frac{1}{2} M \underline{V}_{cm}^2 + \underline{V}_{cm} \cdot \underbrace{\sum m_i \underline{v}_i'}_{=0} + \sum \frac{1}{2} m_i \underline{v}_i'^2$$

$$= T_{cm} + T'$$ ↖ T relative to CM

PE: take system from ① $= \{ \underline{r}_{10}, \underline{r}_{20}, \dots, \underline{r}_{N0} \}$

to ② $= \{ \underline{r}_1 \dots \underline{r}_2 \}$

along some trajectory in $\mathbb{R}^{3N}$.

$$W_{1\to 2} = \underbrace{\sum_{i=1}^{N} \int_{\underline{r}_{i0}}^{\underline{r}_i} d\underline{r}_i \cdot \underline{F}_i}_{A} = \underbrace{\sum_i \int d\underline{r}_i \cdot \underline{F}_i^{ext}}_{B} + \underbrace{\sum_{ij}{}' \int d\underline{r}_i \cdot \underline{F}_{ij}}_{C}$$

$$A = \sum_i \int_1^2 \underline{v}_i \, dt \cdot m_i \frac{d\underline{v}_i}{dt} = \sum_i \int_1^2 m_i \underline{v}_i \cdot d\underline{v}_i$$

$$= \sum_i \frac{1}{2} m \underline{v}_i^2 \Big|_1^2 = T_2 - T_1$$

If $\underline{F}_i^{ext} = -\frac{\partial}{\partial \underline{r}_i} V^{ext}$ is conservative

then $B = -\int_1^2 \sum_i d\underline{r}_i \cdot \frac{\partial V^{ext}}{\partial \underline{r}_i} = -\int_1^2 dV^{ext}$

$$= V(\{\underline{r}_1\}) - V(\{\underline{r}_2\}) = V_1 - V_2$$

If $\underline{F}_{ij} = -\frac{\partial}{\partial \underline{r}_i} V(\underline{r}_i - \underline{r}_j) = +\frac{\partial}{\partial \underline{r}_j} V(\underline{r}_i - \underline{r}_j) = -\underline{F}_{ji}$

also $= -\frac{\partial}{\partial(\underline{r}_i - \underline{r}_j)} V(\underline{r}_i - \underline{r}_j)$

then

$$(C) = \sum_{ij}{}' \int_1^2 d\underline{r}_i \cdot \underline{F}_{ij} = \sum_{ij}{}' \int_1^2 d\underline{r}_j \cdot \underbrace{\underline{F}_{ji}}_{-\underline{F}_{ij}}$$

$$= \frac{1}{2} \sum_{ij}{}' \int_1^2 \underbrace{d(\underline{r}_i - \underline{r}_j)}_{= d\underline{r}_{ij}} \cdot \underbrace{\underline{F}_{ij}}_{\underline{F}_{ij}(\underline{r}_{ij})} \qquad \underline{r}_{ij} \equiv \underline{r}_i - \underline{r}_j$$

$$= -\int_1^2 \frac{1}{2} \sum_{ij}{}' d\underline{r}_{ij} \cdot \frac{\partial V}{\partial \underline{r}_{ij}} = -\int_1^2 \sum_{i<j} d\underline{r}_{ij} \cdot \frac{\partial V}{\partial \underline{r}_{ij}}$$

counts each pair (ij) twice

counts each pair (ij) once

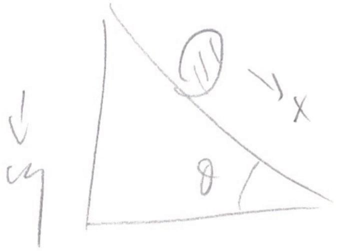
$$= -\int_1^2 dV = V_1 - V_2$$

Combine: $(A) = (B) + (C)$

$$\rightarrow \quad T_2 - T_1 = (V_1^{ext} - V_2^{ext}) + (V_1 - V_2)$$

or $\quad T + V + V^{ext} \equiv E = \text{const}.$

Simple example - sphere rolling (with friction) on an incline



start at $x = \dot{x} = 0$ so $E = 0$

at x: $\Delta y = x \sin\theta$

$$V = -mg\Delta y = -mgx\sin\theta$$

$$T = T_{cm} + T' = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_c\omega^2$$

$\downarrow$ $\frac{2}{5}mR^2$ for a sphere $\qquad \frac{1}{2}mR^2$ for cyl

$\omega = v_{cm}/R$ for rolling $= \dot{x}/R$

$\rightarrow \Delta T = \frac{7}{10}m\dot{x}^2 = -\Delta V$ here $\qquad \frac{3}{4}m\dot{x}^2$ for cyl.

So $\frac{7}{10}m\dot{x}^2 = mgx\sin\theta$ $\qquad$ "

Simpler than direct force/torque balance:

$$N = mg\cos\theta$$

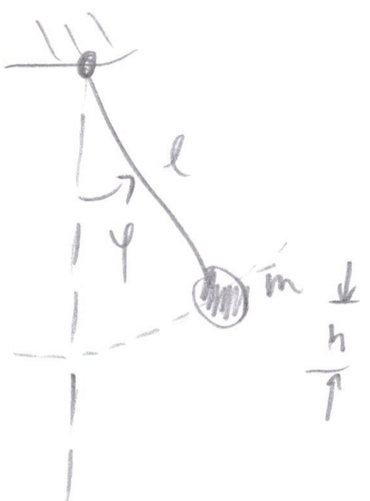
$$m\ddot{x} = mg\sin\theta - f$$

$$I_c\ddot{\varphi} = Rf \qquad \ddot{\varphi} = \ddot{x}/R$$

$$\rightarrow \left(m + I_c/R^2\right)\ddot{x} = mg\sin\theta$$

$$\ddot{x} = \dot{x}\frac{d\dot{x}}{dx} \rightarrow \frac{1}{2}\left(m + I_c/R^2\right)\dot{x}^2 = mg\sin\theta \cdot x$$

Example 2 - pendulum



$T = \frac{1}{2} m (l\dot\varphi)^2$

take $V \equiv 0$ when $h = 0$ at bottom

$\rightarrow V = mgl(1-\cos\varphi)$

$E = \frac{1}{2} ml^2\dot\varphi^2 + mgl(1-\cos\varphi)$

$E = \text{constant} \rightarrow \frac{dE}{dt} = 0 = ml^2\dot\varphi\ddot\varphi + mgl\sin\varphi\,\dot\varphi$

$\sim \ddot\varphi = -\frac{g}{l}\sin\varphi$

— x —

This is a general result for "simple" conservative systems

1-d $\quad E = \frac{1}{2} m v^2 + V(x)$

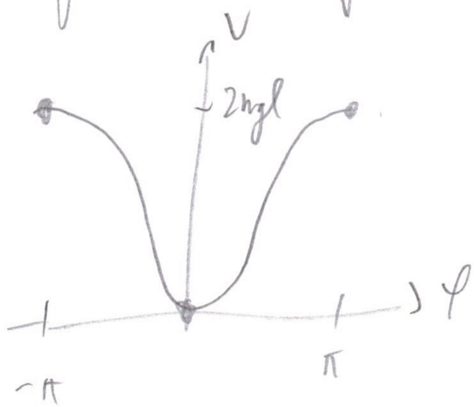
$\frac{dE}{dt} = 0 = m v\dot v + V'(x)\dot x = v\left(m\dot v + V'(x)\right)$

so $m\dot v = -V'(x)$

3-d $\quad E = \frac{1}{2} m \underline{v}^2 + V(\underline{r})$

$\frac{dE}{dt} = 0 = m\underline{v}\cdot\dot{\underline{v}} + \nabla V\cdot\dot{\underline{r}} = (m\dot{\underline{v}} + \nabla V)\cdot\underline{v}$

Equilibria of the pendulum



$V = mgl(1-\cos\varphi)$

$\frac{\partial V}{\partial \varphi} = mgl\sin\varphi = 0$ at $\varphi^* = 0$ or π

: equilibrium points where $f=0$

$\frac{\partial^2 V}{\partial \varphi^2} = mgl\cos\varphi = \begin{cases} + & : \varphi^* = 0 \\ - & : \varphi^* = \pi \end{cases}$

"+" = minimum = stable equil

if $\varphi \lesssim 0$, $-\frac{\partial V}{\partial \varphi} > 0$: restoring force to right

if $\varphi \gtrsim 0$, $-\frac{\partial V}{\partial \varphi} < 0$: restoring force to left

"−" = maximum of V = unstable equilibrium

if $\varphi \lesssim \pi$, $-\frac{\partial V}{\partial \varphi} < 0$: destabilizing force to left

Small oscillations about the equilibrium point:

$\varphi = \varphi^* + \delta\varphi \quad |\delta\varphi| \ll 1$

$\omega^2 = \frac{g}{l}$

$\delta\ddot{\varphi} = -\omega^2 \sin(\varphi^* + \delta\varphi) = -\omega^2 [\underbrace{\sin\varphi^*}_{0} \cos\delta\varphi + \underbrace{\cos\varphi^*}_{\pm 1} \underbrace{\sin\delta\varphi}_{\simeq \delta\varphi}]$

so $\delta\ddot{\varphi} = \mp \omega^2 \delta\varphi$

stable (−) case $\delta\ddot{\varphi} = -\omega^2 \delta\varphi$

$$\delta\varphi(t) = A\cos\omega t + B\sin\omega t$$

= fixed amplitude oscillation

unstable (+) case $\delta\ddot{\varphi} = +\omega^2 \delta\varphi$

$$\delta\varphi(t) = Ce^{\omega t} + De^{-\omega t}$$

= exp. growing disturbance

Motion in general:

$$E = \frac{1}{2} m l^2 \dot{\varphi}^2 + mgl(1-\cos\varphi) = \text{const}$$

so $$\dot{\varphi} = \pm\sqrt{\frac{2}{ml^2}\left(E - mgl(1-\cos\varphi)\right)}$$

$$t = \pm \int d\varphi \Big/ \sqrt{\frac{2}{ml^2}\left(E - \cdots\right)}$$

= "elliptic integral" – tabulated fn.

— X —

3. Taylor 4.37

massless horizontal cylinder rotates

mass M attached to cylinder

mass m on massless string

wrapped around cylinder

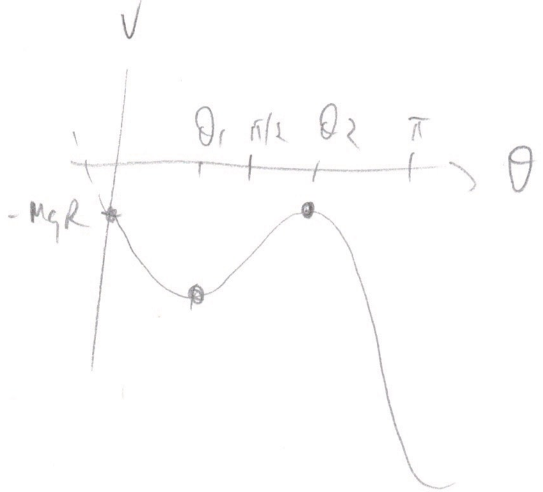
attached to M.

Energy method first:

$$V = Mg\,R(1-\cos\theta) - mgh = ?$$

const string length $l = h + \left(\frac{3\pi}{2} - \theta\right)R + R = \text{const}$

so $V \to V(\theta) = -MgR\cos\theta - mgR\theta + \cancel{\text{const}}$



Equil at

$$\frac{dV}{d\theta} = +MgR\sin\theta - mgR = 0$$

$$\to \sin\theta = \frac{m}{M}$$

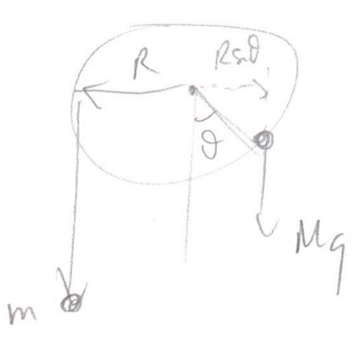
Must have $m < M$ for any equilibrium,

otherwise one side always wins (see later)

2 solutions for $m < M$: $0 < \theta_1 < \frac{\pi}{2}$, $\frac{\pi}{2} < \theta_2 < \pi$

$$\frac{d^2V}{d\theta^2} = MgR\cos\theta = \begin{cases} + & : \theta_1 \text{ stable / min} \\ - & : \theta_2 \text{ unstable / max} \end{cases}$$

Torque method



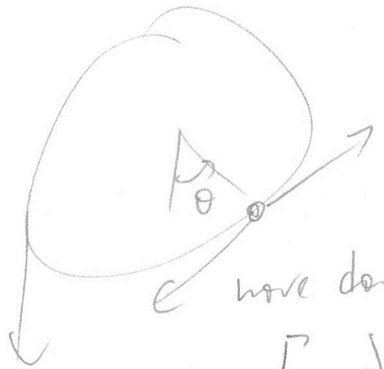
Γ_c = torque abt centre

$= \Gamma_m - \Gamma_M$

$= mgR - MgR\sin\theta$

$= 0$ at equilibm ✓

Stability at θ_1



Γ_m = constant

move up: $\Gamma_M \nearrow$, $\Gamma_c < 0$, $\theta \searrow$

Γ_m = constant

move down: $\Gamma_M \searrow$, $\Gamma_c > 0$, $\theta \nearrow$

So torque generated opposes perturbation → stable

at θ_2

move up $\Gamma_M \searrow$, $\Gamma_c > 0$, $\theta \nearrow$

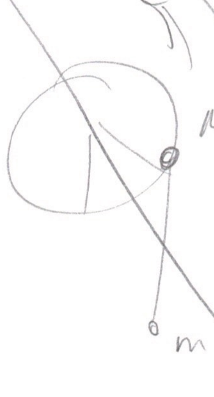
move down $\Gamma_M \nearrow$, $\Gamma_c < 0$, $\theta \searrow$

Torque amplifies perturbation → Unstable

What if $\pi < \theta < \frac{3\pi}{2}$? $\sin\theta < 0$ $\Gamma_c > 0$ always

no equil. pos

~~What if $\theta > 2\pi$?~~



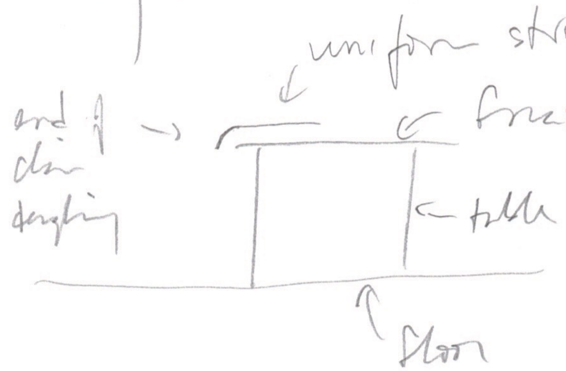
$\cancel{y_m = y_M - l}$

$\cancel{V = MgR(1-\cos\theta) + mg[R(1-\cos\theta) - l]}$

$\cancel{\to (M+m)gR(1-\cos\theta)}$

~~min at $\theta = 2\pi$~~

Falling chain



mass density σ

chain of length L extended on frictionless table

How fast when it leaves the table?

(a) Cons of E method: $E=0$ at start ← table surface is $V=0$

when it leaves $T = \frac{1}{2}(\sigma L)\, v^2$

$V = -(\sigma L)\, g \cdot \frac{L}{2}$ rel to table

total mass

CM position

$V \simeq 0$ on table:

$E_0 = -(\sigma x_0) \cdot g \cdot \frac{x_0}{2} \simeq 0$

if $x_0 \ll L$

⟹

so: $\frac{1}{2}\sigma L v^2 = \frac{1}{2}\sigma L g L$

or $v = \sqrt{gL}$

(Just the speed of a falling object through $L/2$)

(b) $F=ma$ method:

if x is the length over the edge then

$F=ma \;\rightarrow\; g(\sigma x) = \frac{d}{dt}(\sigma L \dot{x})$

or $\ddot{x} = \frac{g}{L}\, x$ or $x = Ae^{\gamma t} + Be^{-\gamma t}$

$\gamma^2 = \frac{g}{L}$

Take $x = \varepsilon$ & $\dot{x} = 0$ at $t = 0$, so

$$\varepsilon = A + B$$

$$0 = \gamma(A - B) \rightarrow A = B = \frac{\varepsilon}{2}$$

$$x(t) = \frac{\varepsilon}{2}\left(e^{\gamma t} + e^{-\gamma t}\right) \simeq \frac{\varepsilon}{2} e^{\gamma t}$$ long enough chain

$x = L$ when $\gamma t^* = \log \frac{2L}{\varepsilon}$: long as $\varepsilon \rightarrow 0$

at which point $\dot{x} = \frac{\varepsilon \gamma}{2} e^{\gamma t^*} = L\gamma = \sqrt{gL}$

c) Variant: what if the chain is in a pile, and only the part hanging over the edge is moving?

use $F = \frac{dp}{dt}$: $(\sigma x) g = \frac{d}{dt}\left(\sigma x \cdot \dot{x}\right)$

or $gx = \dot{x}^2 + x\ddot{x}$: hard to solve

method 1: look for $x = ct^\alpha$ $\rightarrow$

$$gct^\alpha = c^2 \alpha^2 t^{2\alpha - 2} + c^2 \alpha(\alpha - 1) t^{2\alpha - 2}$$

assumes this solution at $t \gg 0$

so need $2\alpha - 2 = \alpha$ or $\alpha = 2$

then $g = 4c + 2c \rightarrow c = \frac{1}{6} g$

method 2: dimensional analysis: $[g] = L/T^2$, $[t] = T$, $[x] = L$

note: σ has dropped out

so $x = cgt^2$, c = numerical constant,

plug in $\rightarrow c = 1/6$

What's the energy when it leaves the table?

$x(t) = \frac{1}{6} g t^2 \to L = \frac{1}{6} g t^{*2}$ or $t^* = \sqrt{6L/g}$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} (\sigma L) \left(\frac{1}{3} g \sqrt{\frac{6L}{g}} \right)^2 = \frac{1}{3} \sigma g L^2$$

$$V = -g \cdot (\sigma L) \cdot \frac{L}{2} = -\frac{1}{2} \sigma g L^2$$

so $E = -\frac{1}{6} \sigma g L^2 \neq 0$

Why? work has to be done against friction to get the chain moving - each link is at rest, then "suddenly" has $v \neq 0$: Δp must be supplied + work done by gravity against friction. $\to$

Generally

$$E = \frac{1}{2} (\sigma x) \dot{x}^2 + \sigma x \cdot g \cdot \left(-\frac{x}{2} \right)$$

$$= \frac{1}{2} \sigma \cdot \frac{g}{6} t^2 \left(\frac{g}{3} t \right)^2 + \sigma \cdot \frac{g t^2}{6} \cdot g \cdot \left(-\frac{g t^2}{12} \right)$$

$$= \sigma g^3 t^4 \underbrace{\left(\frac{1}{2 \times 6 \times 9} - \frac{1}{6 \times 12} \right)}_{\text{negative constant}}$$

$\propto -t^4$!