Energy and Work

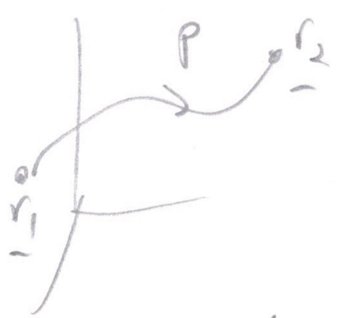
Define kinetic energy $T = \frac{1}{2} m \underline{v}^2 = \underline{p}^2/2m$ for one particle

then $\frac{dT}{dt} = m\, \underline{v} \cdot \dot{\underline{v}} = \underline{F} \cdot \underline{v}$

or $dT = \underline{F} \cdot \underline{v}\, dt = \underline{F} \cdot d\underline{r}$ = work done to move $d\underline{r}$

? intuitively reasonable

If the particle moves from $\underline{r}_1$ to $\underline{r}_2$ along path P



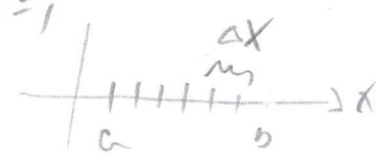
$$\Delta T = \int_{t_1}^{t_2} \underline{F} \cdot \underline{v}\, dt = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r}$$

$= W$: work done to move it

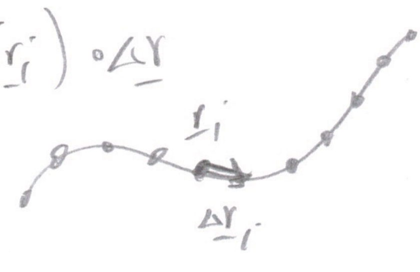
also rate of work done: power $\frac{dW}{dt} = \underline{F} \cdot \underline{v}$

"Line integrals":

usually $\int_a^b dx\, f(x) = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i) \Delta x$



$$\rightarrow \int_{\underline{r}_1}^{\underline{r}_2} d\underline{r} \cdot \underline{F}(\underline{r}) = \lim_{|\Delta \underline{r}| \to 0} \sum \underline{F}(\underline{r}_i) \cdot \Delta \underline{r}$$



Evaluation: $\int_{\underline{r}_1}^{\underline{r}_2} d\underline{r} \cdot \underline{F}(\underline{r}) = \int_{t_1}^{t_2} dt \, \underbrace{\underline{F}(\underline{r}(t)) \cdot \underline{v}(t)}_{\text{function of } t}$

or label P by $y(x), z(x)$

$$\int_{\underline{r}_1}^{\underline{r}_2} d\underline{r} \cdot \underline{F}(\underline{r}) = \int dx \, F_x(x, y(x), z(x)) + \int y'(x) \, dx \, F_y(x, y(x), z(x)) + \int z'(x) \, dx \, F_z(x, y(x), z(x))$$

or label the path by some other variable "s",

e.g. s = arc length

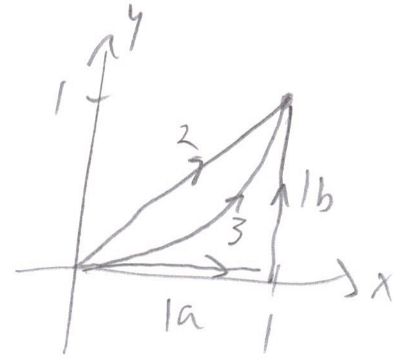
$$\int_{\underline{r}_1}^{\underline{r}_2} d\underline{r} \cdot F(\underline{r}) = \int_{s_1}^{s_2} \underline{r}'(s) \, ds \cdot F(x(s), y(s), z(s))$$

All of these are the same - change of integration variable, choice by convenience!

Key question: does the line integral depend on P or just $\underline{r}_1$ & $\underline{r}_2$?

Example $\underline{F} = (2xy, x^2)$ in 2d

along paths 1, 2, 3



(1) $x: 0 \to 1$ at $y=0$, then $y: 0 \to 1$ at $x=1$

$$I_{1a} = \int_{x=0}^{1} (dx, 0) \cdot (0, x^2) = 0$$

$$I_{1b} = \int_{y=0}^{1} (0, dy) \cdot (2y, 1) = \int_0^1 dy = 1$$

(2) $x = y$, $x: 0 \to 1$

$$I_2 = \int_{x=0}^{1} (dx, dx) \cdot (2x^2, x^2) = \int_0^1 dx\, 3x^2 = 1$$

(3) $y = x^2$ $x: 0 \to 1$

$$I_3 = \int_0^1 dx\, (dx, 2x\,dx) \cdot (2x^3, x^2) = \int_0^1 dx\, 4x^3 = 1$$

Book does $\underline{F} = (y, 2x)$ with same paths

$\to$ different results for each P

Distinction: does $\underline{F} = -\nabla V$ $V = V(\underline{r})$?

$\underline{F} = (2xy, x^2) \to V = -x^2 y$

$\underline{F} = (y, 2x) \to$ no!

why? $\underline{F} = (y, 2x) = -\nabla V \rightarrow \frac{\partial V}{\partial x} = -y \qquad \frac{\partial V}{\partial y} = -2x$

?? $V = -xy + f(y) \qquad V = -2xy + f(x)$

When V exists → "potential" for $\underline{F}$

- sign: convention $\underline{F} = -\nabla V$ means force drives particle from high V to low V

cf. $\begin{cases} \text{pressure: drives fluid from high } p \text{ to low } p \\ \text{temperature heat goes from high to low } T \end{cases}$

General argument: $\underline{F} = -\nabla V = \int_P d\underline{r} \cdot \underline{F}(\underline{r})$ indep. of P

$$\int_P \underline{F} \cdot d\underline{r} = \sum_i \underline{F}(\underline{r}_i) \cdot \Delta \underline{r}_i$$

$$\rightarrow -\sum_i \nabla V(\underline{r}_i) \cdot \Delta \underline{r}_i$$

look at the Taylor series for V:

$$V(\underline{r} + \Delta \underline{r}) = V(\underline{r}) + \Delta \underline{r} \cdot V(\underline{r}) + O(\Delta \underline{r}^2)$$

so $$\int \underline{F} \cdot d\underline{r} = -\sum_i \left[V(\underline{r}_{i+1}) - V(\underline{r}_i) \right]$$

$$= -\left[V(\underline{r}_N) - V(\underline{r}_1) \right]$$

endpoints of P

In mechanics, when $\underline{F} = -\nabla V$

$$W(\underline{r}_1 \to \underline{r}_2) = T(\underline{r}_2) - T(\underline{r}_1) = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}\cdot d\underline{r} = -\left[V(\underline{r}_2) - V(\underline{r}_1)\right]$$

or $$T(\underline{r}_1) + V(\underline{r}_1) = T(\underline{r}_2) + V(\underline{r}_2)$$

This works for any $\underline{r}_1$ + $\underline{r}_2$ so $E = T + V = \text{constant}$

total energy, kinetic, potential energy

Usually fails if $V = V(\underline{r}, t)$:

$$\int_{\underline{r}_1}^{\underline{r}_2} \underline{F}\cdot d\underline{r} = -\int_{\underline{r}_1}^{\underline{r}_2} \left[\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy\right]$$

$$= -\int_1^2 \left[dV - \frac{\partial V}{\partial t}dt\right]$$

$$= V(\underline{r}_1) - V(\underline{r}_2) + \text{extra term}$$

$\to$ E not necessarily conserved here.

When does $\underline{F} = -\nabla V$?

gravity $\underline{F} = m\underline{g} \to V = -m\underline{g}\cdot\underline{r}$

electrostatics $F = \frac{c}{r^3}\underline{r} \to V = -\nabla\left(\frac{c}{r}\right)$

(check: $\nabla \frac{1}{r} = \nabla\left(x^2+y^2+z^2\right)^{-1/2} = \frac{-x}{(\;)^{3/2}}\hat{x} + \dots = -\underline{r}/r^3$)

central forces; $\underset{\sim}{F} = \underline{r}\,\varphi(|\underline{r}|) \rightarrow V = -\int_{r_0}^{r} dr'\, r' \varphi(r')$

proof: $\frac{\partial}{\partial r} \int_{r_0}^{g(r)} dr'\, f(r', r)$

$$= \lim_{\Delta r \to 0} \frac{1}{\Delta r} \left[\int_{r_0}^{g(r+\Delta r)} dr'\, f(r', r+\Delta r) - \int_{r_0}^{g(r)} dr'\, f(r', r) \right]$$

$$= \lim \frac{1}{\Delta r} \left[\int_{r_0}^{g(r)+g'(r)\Delta r} dr' \left(f(r, r') + \frac{\partial f}{\partial r} \Delta r \right) - \int_{r_0}^{g(r)} dr'\, f(r', r) \right]$$

$$= \lim \frac{1}{\Delta r} \left[\int_{g(r)}^{g(r)+g'(r)\Delta r} dr'\, f(r', r) + \int_{r_0}^{g(r)} dr' \frac{\partial f(r', r)}{\partial r} \Delta r + O(\Delta r^2) \right]$$

$$= g'(r)\Delta r \cdot f(g(r), r)$$

$$= g'(r)\, f(g(r), r) + \int_{r_0}^{g(r)} dr' \frac{\partial f(r', r)}{\partial r}$$

Here $g(r) \rightarrow r$; $\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} = \frac{x}{r} \frac{\partial}{\partial r}$ or $\nabla = \hat{r} \frac{\partial}{\partial r}$

$f = r' \varphi(r')$

so $-\nabla V = +\hat{r}\left(r \varphi(r) \right) = \underline{r}\, \varphi(\underline{r})$

Conservative systems:

the following statements are equivalent

1. $\underline{F} = -\nabla V$: potential energy exists, $E \equiv T + V = \text{constant}$

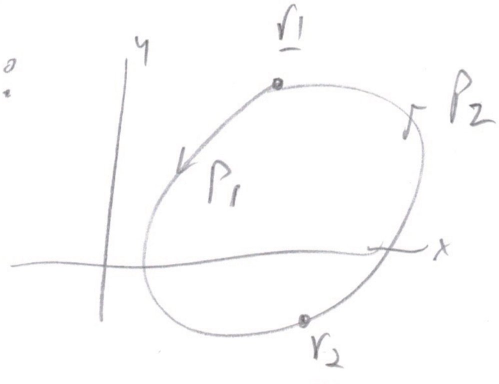
2. $\nabla \times \underline{F} = 0$

3. $\int_1^2 d\underline{r} \cdot \underline{F}$ is independent of the path from 1 to 2

3a. $\oint \underline{F} \cdot d\underline{r} = 0$ for a closed loop

proof:

also works in 3d



loop $= P_1 + P_2$

$$\oint = \int_{P_1} + \int_{P_2}$$

$$= \int_{P_1} - \int_{(-P_2)} = 0$$

2 different paths $\underline{r_1} \to \underline{r_2}$

so $3 \Leftrightarrow 3a$

$1 \Rightarrow 3$ proved above

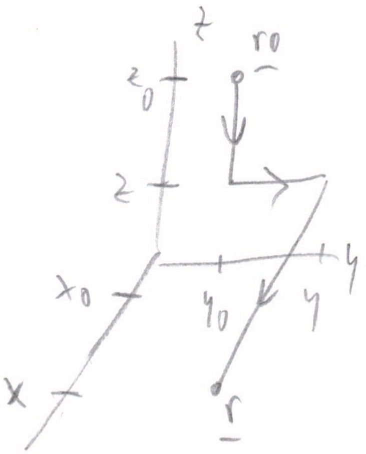
$1 \Rightarrow 2$ trivial

$2 \Leftrightarrow 3a$ use Stokes' Thm - if S is an open surface bounded by curve ∂S

then $$\int_S d\underline{A} \cdot \nabla \times \underline{F} = \oint_{\partial S} d\underline{r} \cdot \underline{F} = 0$$

for any loop ∂S or open surface S.

$3 \Rightarrow 1$: construct a candidate V from $\underline{F}$ & show it works



$$\text{claim} \quad V(\underline{r}) = -\int_{x_0}^{x} dx'\, F_x(x', y, z) - \int_{y_0}^{y} dy'\, F_y(x_0, y', z) - \int_{z_0}^{z} dz'\, F_z(x_0, y_0, z')$$

take $V(\underline{r_0}) = 0$

then $\dfrac{\partial V}{\partial x} = -F_x(x,y,z)$ ✓

$$\frac{\partial V}{\partial y} = -\int_{x_0}^{x} dx' \underbrace{\frac{\partial F_x(x',y,z)}{\partial y}}_{\substack{\shortparallel \\ \frac{\partial F_y}{\partial x}(x',y,z)}} - F_y(x_0,y,z)$$

$$= -\left[F_y(x,y,z) - \cancel{F_y(x_0,y,z)} \right] - \cancel{F_y(x_0,y,z)}$$

$$\frac{\partial V}{\partial z} = -\int_{x_0}^{x} dx' \underbrace{\frac{\partial F_x(x',y,z)}{\partial z}}_{\substack{\shortparallel \\ \frac{\partial F_z}{\partial x}(x',y,z)}} - \int_{y_0}^{y} dy' \underbrace{\frac{\partial F_y(x_0,y',z)}{\partial z}}_{\substack{\shortparallel \\ \frac{\partial F_z}{\partial y}(x_0,y',z)}} - F_z(x_0,y_0,z)$$

$$= -\left[F_z(x,y,z) - \cancel{F_z(x_0,y,z)} \right] - \left[\cancel{F_z(x_0,y,z)} - \cancel{F_z(x_0,y_0,z)} \right] - \cancel{F(x_0,y_0,z)}$$

$$= -F_z(x,y,z)$$

i.e. $\nabla V = -\underline{F}$ provided $\nabla \times \underline{F} = 0$

If V is known, $\underline{F} = -\nabla V$ follows immediately

If $\underline{F}$ is known, $V = ?$

(1) Use line integral $V = -\int_{\underline{r}_0}^{\underline{r}} d\underline{r} \cdot \underline{F}$

must choose a path where integral can be done

(2) "Trial + error" - integrate + match

e.g. $\underline{F} = (2xy, x^2, 0)$ as above

check $\nabla \times \underline{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2 & 0 \end{vmatrix}$

$= \hat{x}(0) + \hat{y}(0) + \hat{z}(2x - 2x)$

$= 0$

then

$F_x = 2xy = -\frac{\partial V}{\partial x} \rightarrow V = -x^2 y + f(y, z)$

$F_y = x^2 = -\frac{\partial V}{\partial y} \rightarrow V = -x^2 y + g(x, z)$

$F_z = 0 = -\frac{\partial V}{\partial z} \rightarrow V = h(x, y)$

consistent w/ $f = g = 0$, $h = -x^2 y$.

What about the $\underline{r_0}$ in $V(\underline{r}) = -\int_{\underline{r_0}}^{\underline{r}} d\underline{r}' \cdot \underline{F}(\underline{r}')$?

changing $\underline{r_0}$ to $\underline{r_0}'$ changes V by $\int_{\underline{r_0}}^{\underline{r_0}'} d\underline{r} \cdot \underline{F} =$ constant

which changes E by the same constant, but drops out

of $m\ddot{\underline{r}} = -\nabla V$

$\Rightarrow$ choose $V = 0$ at any convenient $\underline{r_0}$.

Non-conservative forces - often friction

in air $m\dot{\underline{v}} = -\nabla V - \underline{v}\, f(v)$ $\quad f = \begin{cases} \text{constant - low } v \\ c_D v \text{ - large } v \end{cases}$

so $\dfrac{dE}{dt} = \dfrac{d}{dt}\left(\frac{1}{2}m\underline{v}^2 + V\right)$

$= m\underline{v}\cdot\dot{\underline{v}} + \nabla V \cdot \dot{\underline{r}} = (m\dot{\underline{v}} + \nabla V)\cdot\underline{v}$

$= -\underline{v}^2 f(v) < 0$

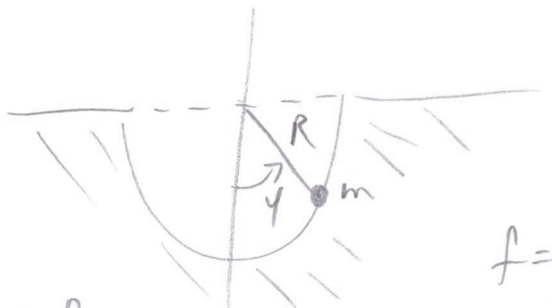
physically - energy is not conserved because some of it is dissipated into heat.

sliding friction:

$$\underline{f} = -\left(\text{positive expression}\right)\hat{v}$$

$$\underline{f}\cdot\underline{v} = -\left(\quad ''\quad\right) v < 0 \quad \text{: energy lost}$$

example



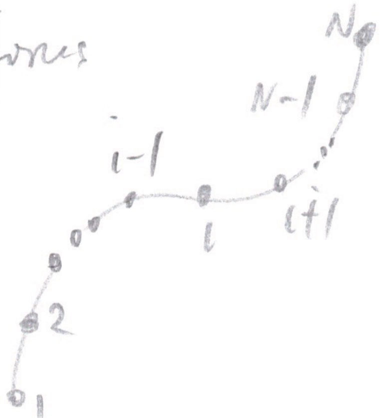
$$f = \mu N = \mu m g \cos\varphi$$

$$\rightarrow \underline{f} = \pm \mu m g \cos\varphi\,\hat{\varphi}$$

$$\underline{\nabla}\times\underline{f} = \hat{z}\,\frac{1}{r}\frac{\partial}{\partial r}\left(r f_\varphi\right) \neq 0$$

use formula inside front cover / Taylor

"chiral" forces



chain molecule

$$\underline{F}_i = k\left(\underline{r}_i - \underline{r}_{i-1}\right)\times\left(\underline{r}_{i+1} - \underline{r}_i\right)$$

$$\frac{\partial}{\partial \underline{r}_i}\times\underline{F}_i = \cdots \neq 0$$