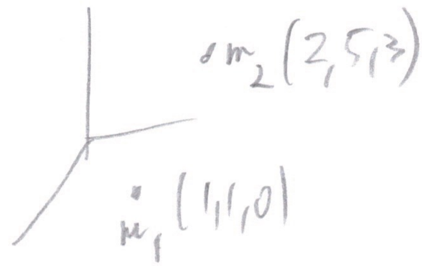
What is I? $I \equiv \sum m_i r_i^2 = ?$

mass points m_1 at $\underline{r}_1$, m_2 at $\underline{r}_2$ ... just add them up

e.g.



$$I = m_1(1^2+1^2) + m_2(2^2+5^2) \ldots$$

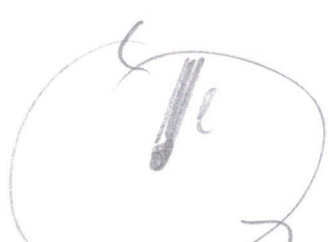
continuous bodies $m_i = \rho(\underline{r}_i)\, dV$ ρ = density

$$M = \sum m_i = \sum \rho(\underline{r}_i)\, dV \rightarrow \int dV \rho(\underline{r})$$

$$I = \sum m_i \sigma_i^2 = \sum \rho(\underline{r}_i)\, dV \sigma^2(r_i) \rightarrow \int dV \rho(\underline{r})\, \sigma^2(\underline{r})$$

which is a calculus problem

Ex1: rod rotating about an end in a plane



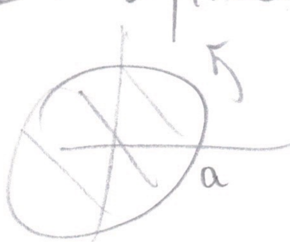
s = coordinate along rod

λ = mass/length

$$I = \int_0^l (\lambda\, ds)\, s^2 = \lambda l^3/3$$

$$\rightarrow \frac{1}{3} M l^2, \quad M = \lambda l$$

Ex2: cylinder rotating about its axis



$$I = \int d^3r\, \rho\, \sigma^2$$

$$\rightarrow \int_0^{2\pi} d\varphi \int_0^a \sigma\, d\sigma \int_0^L dz \cdot \rho\, \sigma^2$$

$$= 2\pi \cdot \frac{a^4}{4} \cdot L \cdot \rho = \frac{1}{2}(\rho \pi a^2 L)\, a^2 = \frac{1}{2} M a^2$$

Ex3 sphere rotating about a diameter

$$I = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^{a} r^2 dr \cdot \rho \cdot \underbrace{(x^2+y^2)}_{r^2 \sin^2\theta}$$

$$= 2\pi \cdot \frac{4}{3} \cdot \frac{a^5}{5} \cdot \rho = \frac{2}{5}\left(\frac{4}{3}\pi a^3 \rho\right) \cdot a^2$$

$$= \frac{2}{5} M a^2$$

— ~ —

What about a cylinder rotating about a point on circumference?

very messy calculus problem

Instead use Parallel Axis Theorem:

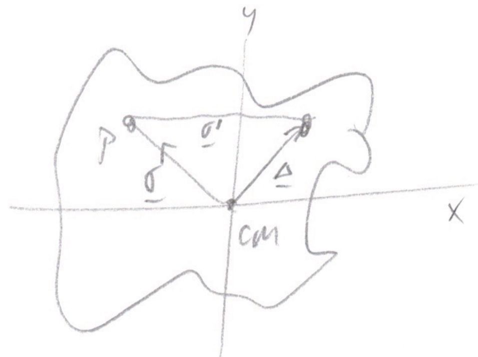
I_0 = moment of inertia about CM

I_Δ = " " about ∥ axis displaced by Δ in plane

then $I_\Delta = I_0 + M\Delta^2$

proof: take CM at origin, $\hat{z}$ = rotation axis

at some z



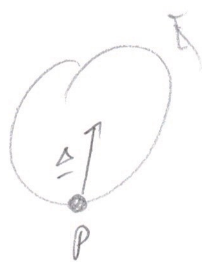
$$I_{cm} = \int d^3r \, \rho \, \sigma^2$$

$$\underline{R}_{cm} = \frac{1}{M}\int d^3r \, \rho \, (\sigma_x, \sigma_y, z) = 0$$

$$I_\Delta = \int d^3r \, \rho \, \sigma'^2 = \int d^3r \, \rho \, |\underline{\sigma} - \underline{\Delta}|^2$$

$$= \underbrace{\int d^3r \, \rho \, \sigma^2}_{I_0} - 2\underline{\Delta}\cdot \underbrace{\int d^3r \, \rho \, \underline{\sigma}}_{0} + \underline{\Delta}^2 \underbrace{\int d^3r \, \rho}_{M}$$

So for

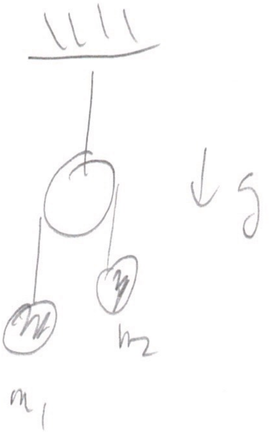


$$I_P = I_{cm} + M\Delta^2$$

$$= \frac{1}{2}Ma^2 + M\cdot a^2 = \frac{3}{2}Ma^2$$

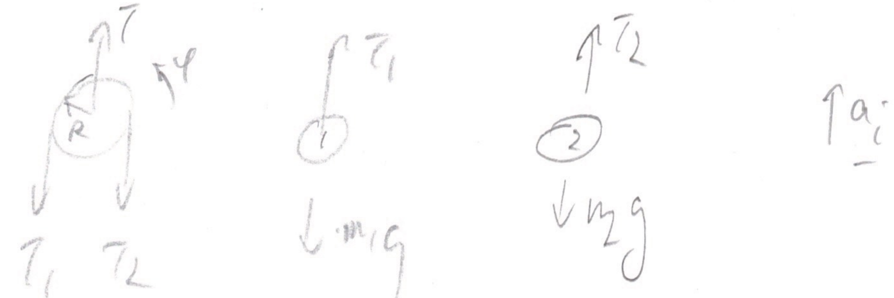
I_Δ is bigger because the mass is farther from the rotation axis.

Example - Atwood's machine



{ massless, unstretchable string | no slipping
("?) frictionless pulley

vertical force balance



If $I=0$:

$m_1 a_1 = T_1 - m_1 g$

$m_2 a_2 = T_2 - m_2 g$

torque balance on pulley

$(T_1 - T_2)\cdot R = I\ddot{\varphi} \to 0$

$\to T_1 = T_2$

unstretchable $\to a_1 = -a_2 \to$

$m_1 a_1 = T - m_1 g$

$-m_2 a_1 = T - m_2 g$

$$a_1 = \frac{m_2 - m_1}{m_2 + m_1} g$$

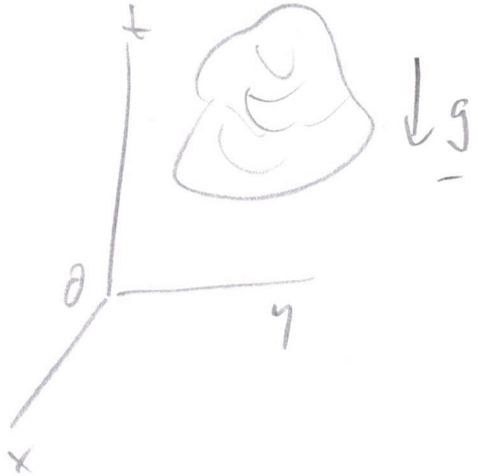
If $I \neq 0$: $R\ddot{\varphi} = -a_1$, $T_1 - T_2 = -I/R^2 \cdot a_1 \xrightarrow{\text{cyl.}} -\frac{1}{2} M a_1$

{ This uses $\frac{dL}{dt} = \Gamma$, L, Γ wrt fixed center of pulley. }

$$\ldots\ a_1 = \frac{m_2 - m_1}{m_2 + m_1 + \frac{1}{2}M} g$$

General remark:

gravitational forces + torques "act at the CM"



$$\underline{F}_{grav} = \sum_i m_i \underline{g} = M\underline{g}$$

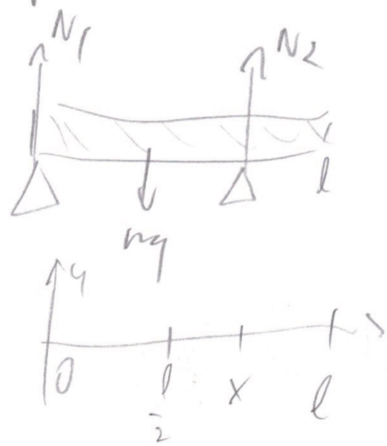
$$\underline{\Gamma}_{grav} = \sum_i \underline{r}_i \times (m_i \underline{g})$$

$$= \left(\sum_i m_i \underline{r}_i\right) \times \underline{g}$$

$$= \underline{R}_{cm} \times M\underline{g}$$

The shape of the object enters via $\underline{I}$.

Example: solid bar, length l, supported at 2 points: 0 + x



force balance: $N_1 + N_2 = mg$

if $\frac{l}{2} < x < l$: $N_1 + N_2 = mg$

pos. direction for Γ, ω

torque about supports
$$\Gamma_0 = -mg\cdot\frac{l}{2} + N_2 x = 0$$
$$\Gamma_x = mg\left(x - \frac{l}{2}\right) - N_1 x = 0$$

so $N_1 = mg\left(1 - \frac{l}{2x}\right)$, $N_2 = mg\cdot\frac{l}{2x}$; $N_1 + N_2 = mg$ ✓

Torque about some other point y:

$$\Gamma_y = -N_1 \cdot y + mg\cdot\left(y - \frac{l}{2}\right) + N_2(x - y)$$
$$= -mg\left(\cancel{y} - \cancel{\frac{l}{2}\frac{y}{x}}\right) + mg\left(\cancel{y} - \cancel{\frac{l}{2}}\right) + \cancel{mg\frac{l}{2}} - \cancel{mg\frac{l}{2}\frac{y}{x}} = 0$$

$\Rightarrow$

What if $0 < x < \frac{l}{2}$?

$$\Gamma_x = -N_1 x - mg\left(\frac{l}{2} - x\right) < 0 \quad : \text{must rotate.}$$

Remove support 2; $N_1 = ?$

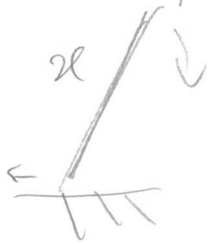
about 1

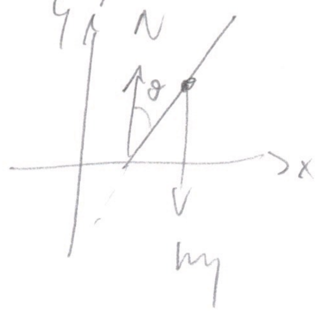
$$\Gamma_1 = I_1 \alpha = -mg\,l/2 \quad \text{so} \quad \alpha = -\frac{3g}{2l} \to a_{cm} = \frac{l}{2}\alpha = -\frac{3g}{4}$$

$I_1 = \frac{1}{3}ml^2$

$$N_1 - mg = ma \to N_1 = \frac{mg}{4}$$

Example: pencil tipping over on a frictionless table





θ = tip angle

(x,y) = CM coordinates

$$m\ddot{x} = 0 \quad \text{so} \quad \dot{x} = \text{const} \to 0 \text{ if starts at rest.}$$

$$m\ddot{y} = N - mg$$

$$\dot{L}_{cm} = \Gamma_{cm} \quad \text{or} \quad I_{cm}\ddot{\theta} = +N\ell\sin(\pi-\theta) = N\ell\sin\theta$$

geometry: $y = \ell\cos\theta$ so $\dot{y} = -\ell\sin\theta\,\dot{\theta}$

$$\ddot{y} = -\ell\left(\sin\theta\,\ddot{\theta} + \dot{\theta}^2\cos\theta\right)$$

$\parallel$ axis thm: $I_{cm} = \underbrace{\frac{1}{3}M\ell^2}_{I_{tip}} - M(\ell/2)^2 = \frac{1}{12}M\ell^2$

combine

$$I_{cm}\ddot{\theta} = (mg + m\ddot{y})\,\ell\sin\theta$$

$$= m\ell\sin\theta\left(g - \ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\cos\theta\right)$$

$$\text{or} \quad \ddot{\theta} = \sin\theta\left(g - \dot{\theta}^2\ell\cos\theta\right) \Big/ \left(\tfrac{1}{12} + \sin^2\theta\right)\ell$$

Add friction: $m\ddot{x} = \mu_s N$ (+ to prevent sliding to left)

$$m\ddot{y} = N - mg$$

$$I_{cm}\ddot{\theta} = N\ell\sin\theta - \mu_s N\ell\cos\theta$$

etc.

For small θ: $\ddot{\theta} \simeq \frac{12g}{\ell}\sin\theta + \ldots$: unstable

Subtlety: the derivation of $\dot{\underline{L}} = \underline{\Gamma}$ assumed that $\underline{L}$ & $\underline{\Gamma}$, were taken wrt some fixed point in space

In Atwood problem & 2nd method above this is true, but in 1st method used the CM which moves (accelerates!)
+ pencil problem

Is this OK?

Take an arbitrary point $\underline{\xi}(t)$ to define $\underline{L}$ & $\underline{\Gamma}$

$$\underline{L}_{\underline{\xi}} \equiv \sum_i (\underline{r}_i - \underline{\xi}) \times m_i (\dot{\underline{r}}_i - \dot{\underline{\xi}})$$

$\underline{r}_i$ relative to fixed pt.

$$= \sum_i \underline{r}_i \times m_i \dot{\underline{r}}_i - \underline{\xi} \times M\dot{\underline{R}} - M\underline{R} \times \dot{\underline{\xi}} + M \underline{\xi} \times \dot{\underline{\xi}}$$

$\therefore$ $\underline{L}$ depends on choice of origin

$$\frac{d}{dt}\underline{L}_{\underline{\xi}} = \sum_i (\underline{r}_i - \underline{\xi}) \times m_i (\ddot{\underline{r}}_i - \ddot{\underline{\xi}})$$

$$= \underbrace{\sum_i (\underline{r}_i - \underline{\xi}) \times \underline{F}_i}_{\underline{\Gamma}_{\underline{\xi}}} \underbrace{- M\underline{R} \times \ddot{\underline{\xi}} + M \underline{\xi} \times \ddot{\underline{\xi}}}_{\text{extra terms}}$$

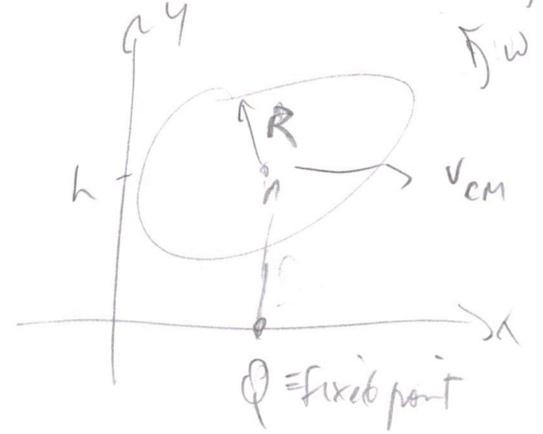
Extra terms $\to 0$ if (1) $\underline{\xi} = \underline{R} =$ CM position

$$\dot{\underline{L}}_{CM} = \underline{\Gamma}_{CM}, \quad \underline{L}_{CM} = \underline{L}_{\underline{0}} - \underline{R} \times \underline{P}$$

or (2) $\ddot{\underline{\xi}} = 0 \to \dot{\underline{L}}_{\underline{\xi}} = \underline{\Gamma}_{\underline{\xi}}$

Disc translating + rotating on a frictionless plane

(1)



Angular momentum about Q?

Use previous with origin → Q

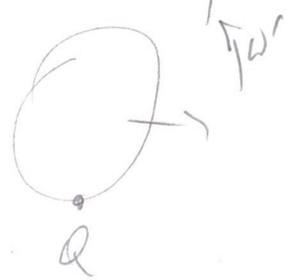
$\underline{L}_{cm} = \underline{L}_Q - \underline{R} \times \underline{P}$

$I_c \omega \hat{z}$ $\qquad -(h\hat{y}) \times (m v_c \hat{x})$

$= + m v_c h \hat{z}$

So $\quad \underline{L}_Q = (I_c \omega - m v_c h)\hat{z}$

(2) Disc suddenly fixed in place at Q; motion?



starts to rotate about Q; ω_Q = ?

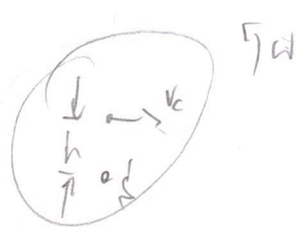
No torque about Q so $L_Q = L_Q'$

$I_c \omega - m v_{cm} \underset{\substack{\uparrow \\ h = R \text{ at } Q}}{R} = (I_c + \underset{\substack{\uparrow \\ \text{‖-axis thm}}}{mR^2})\,\omega'$

$I_c = \frac{1}{2} MR^2$

so $\quad \omega' = \dfrac{\frac{1}{2} MR^2 \omega - m v_{cm} R}{\frac{3}{2} mR^2} = \dfrac{R\omega - 2v_{cm}}{3R}$

(3) At what point S would the disc not rotate

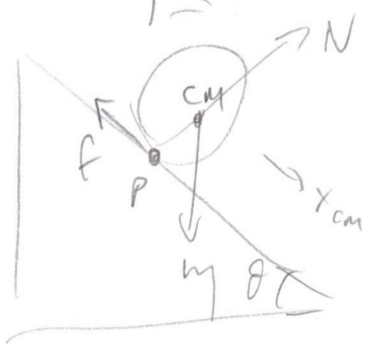


Want $L_S' = 0 = I_c \omega - m v_c h = 0$

$\therefore h = R^2\omega / 2v_c$

(provided $h < R$!)

Sphere rolling down an inclined plane: (3.55)



notice f opposes motion at contact pt.

pure rolling $x_{cm} = R\varphi$

φ = ∢ rolled through

1. use $\dot{\underline{L}}_{cm} = \underline{\Gamma}_{cm}$: $I_{cm}\ddot{\varphi} = I_{cm}\left(a_{cm}/R\right) = fR$

$$ma_{cm} = mg\sin\theta - f$$

eliminate f

$$\left(-ma_{cm} + mg\sin\theta\right)R = I_{cm}\,a_{cm}/R$$

$$a_{cm} = \frac{mg\sin\theta\, R}{mR + I_c/R} = \frac{mgR^2\sin\theta}{I_c + mR^2}$$

2. use $\dot{\underline{L}}_P = \Gamma_P$: $\underbrace{I_P}_{= I_c + mR^2}\left(a_c/R\right) = mgR\underbrace{\sin(\pi-\theta)}_{=\sin\theta}$

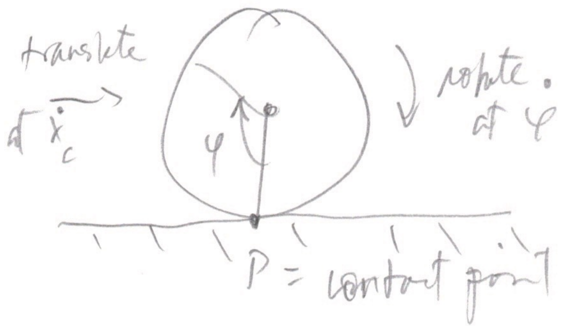
∴ same thing

If sphere placed at rest – friction holds contact point → rotates

If thrown down – contact point slides at first → (later)

Rolling, sliding & slipping

ball (sphere or cyl.) moving across a plane surface



$x(t)$ = CM position

$\varphi(t)$ = angle rotated through at t

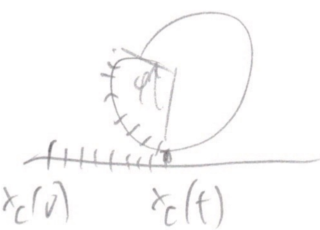
velocity of P relative to CM is $-R\dot\varphi$

so $\dot x_p = \dot x_c - R\dot\varphi$

pure sliding : whole ball moves at $\dot x_c$, $\dot\varphi = 0$, $\dot x_p = \dot x_c$

pure rolling : $\dot x_p = 0$ $\quad \dot x = R\dot\varphi$

(other argument :



horizontal disp = $\dot x_c \Delta t$

surface disp = $R\dot\varphi\Delta t$

these are =)

intermediate case "slipping"

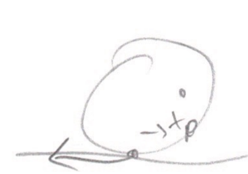
$\dot x_p > 0 \quad \dot x > R\dot\varphi > 0$

friction : always opposes motion → opposite $\dot x_p$

rolling

$\underline{f_s} = \mu_s N \hat{x}$

slipping/sliding

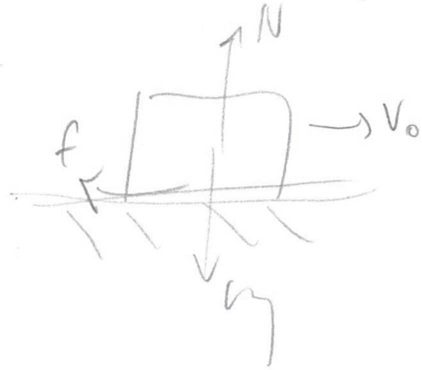


$\underline{f} = -\mu_k N \hat{x}$

→ slipping tends to be transient

Transition from sliding to rolling

simpler version



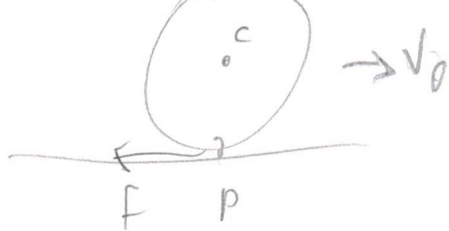
block set down at v_0

$ma = -f = -\mu m g$

$v(t) = v_0 - \mu g t$

slows → $v = 0$ at $t = v_0/\mu g$

disc



$a = -\mu g$ again

$I\dot{\omega}_c = fR \rightarrow \dot{\omega}_c = -2\mu g/R$

$\omega_c(t) = -\frac{2\mu g}{R} t$

$V_{cm} = v_0 - \mu g t$ again

$V_p = V_{cm} + R\omega = v_0 - 3\mu g t \rightarrow 0$

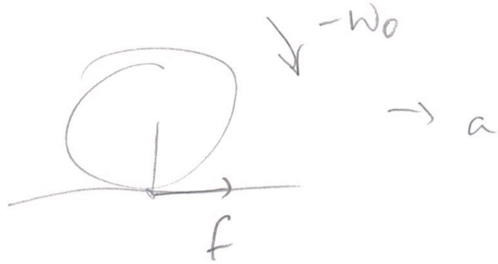
at $t = v_0/3\mu g$

at which point $V_{cm} = \frac{2}{3} v_0$

$\omega = -2v_0/3R = -V_{cm}/R$

and rolling starts.

Same thing with initial angular velocity:



friction opposes relative motion at contact pt.

Start: $ma = f = \mu m g$ again

$$I_c \dot{w}_c = fR \to \dot{w}_c = 2\mu g/R$$

After $v_{cm} = 0 + at = \mu g t$

$$V_p = v_{cm} + Rw = \mu g t + R(-w_0 + 2\mu g t/R)$$

$$= 3\mu g t - Rw_0 \to 0 \text{ at } t = Rw_0/3\mu g$$

where $V_{cm} = Rw_0/3$

$$w_c = -w_0 + \frac{2\mu g}{R} \cdot \frac{Rw_0}{3\mu g} = -\frac{1}{3}w_0$$

and $v_{cm} = -Rw_c$ here.