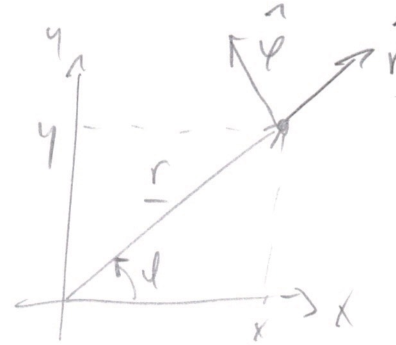
Rotation kinematics – 2d, single rotation axis

polar coordinates



$$\begin{cases} r = \sqrt{x^2+y^2} \\ \tan\varphi = y/x \end{cases}$$

or $$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

introduce local coordinate system $(\hat{r}, \hat{\varphi})$ at each $\underline{r}$

where $\hat{r}$ in direction of increasing r

$\hat{\varphi}$ " " " φ

$\to$ $\hat{r} = \underline{r}/r = (\cos\varphi, \sin\varphi)$

if $\hat{\varphi} = (a, b)$ then $\hat{\varphi}\cdot\hat{r} = 0$, $\hat{\varphi}^2 = 1$

or $$\left.\begin{array}{l} a\cos\varphi + b\sin\varphi = 0 \\ a^2+b^2 = 1 \end{array}\right\} \to \begin{array}{l} a = -\sin\varphi \\ b = \cos\varphi \end{array}$$

and $\hat{\varphi} = (-\sin\varphi, \cos\varphi)$

As $\underline{r}(t)$ moves $\hat{r}$ + $\hat{\varphi}$ rotate

$$\frac{d\hat{r}}{dt} = (-\sin\varphi\,\dot{\varphi}, \cos\varphi\,\dot{\varphi}) = \hat{\varphi}\,\dot{\varphi}$$

$$\frac{d\hat{\varphi}}{dt} = (-\cos\varphi\,\dot{\varphi}, -\sin\varphi\,\dot{\varphi}) = -\hat{r}\,\dot{\varphi}$$

so $\frac{d\underline{r}}{dt} = \frac{d}{dt}\left(r\hat{r}\right) = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$ is the velocity

Acceleration is

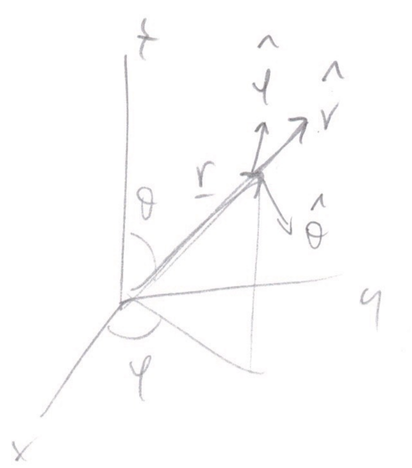
$$\underline{a} = \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) = \left(\ddot{r}\hat{r} + \dot{r}\dot{\varphi}\hat{\varphi}\right) + \left(\dot{r}\dot{\varphi}\hat{\varphi} + r\ddot{\varphi}\hat{\varphi} - r\dot{\varphi}^2\hat{r}\right)$$

$$= \hat{r}\left(\ddot{r} - r\dot{\varphi}^2\right) + \hat{\varphi}\left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right)$$

So if a force $\underline{F} = \hat{r}F_r + \hat{\varphi}F_\varphi$ is applied,

$$F_r = m\left(\ddot{r} - r\dot{\varphi}^2\right) \qquad F_\varphi = m\left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right)$$

What about 3d?



$\hat{r}, \hat{\theta}, \hat{\varphi}$ in direction of increasing r, φ, θ

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$\ddot{\underline{r}}$ = mess.

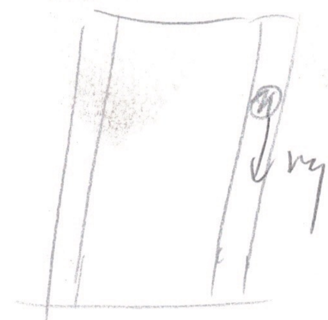
Notice that if r = const $\quad F_r = -mr^2\dot{\varphi}^2$: centripetal force

$F_\varphi = mr\ddot{\varphi}$ angular force

1.49 Small sphere in narrow gap btw cylinders, no friction

top :

side :



$$m(\ddot{r} - r\dot{\varphi}^2) = N \xrightarrow{r=R} -mR\dot{\varphi}^2$$

$$m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) = 0 \rightarrow \ddot{\varphi} = 0$$

$$m\ddot{z} = -mg$$

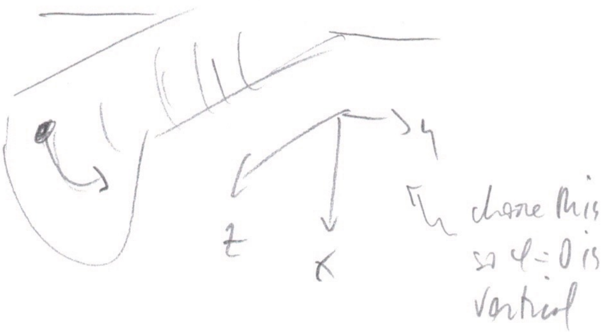
so $\varphi = \varphi_0 + \omega_0 t$ $\quad \varphi_0, \omega_0 =$ constants

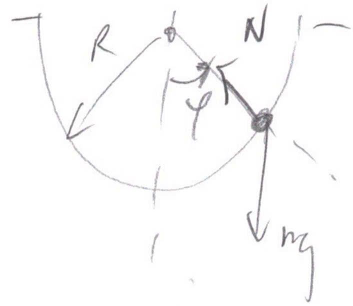
$$z = z_0 + \dot{z}_0 t - \tfrac{1}{2} g t^2$$

$N = -mR\omega_0^2$ centripetal force

with friction? sphere will rotate as it falls
need more technology
(w/o friction sphere keeps its initial $\underline{\Omega}$)

Ex: skateboard in a frictionless half-pipe:





$$F_r = mg\cos\varphi - N = m(\ddot{r} - r\dot{\varphi}^2) \to -mR\dot{\varphi}^2$$

$$F_\varphi = -mg\sin\varphi = m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) \to mR\ddot{\varphi}$$

$$F_z = 0 = m\ddot{z}$$

So $z = z_0 + \dot{z}_0 t$: linear force-free motion

$$\boxed{\ddot{\varphi} = -\frac{g}{R}\sin\varphi}$$

$$N = mg\cos\varphi + mR\dot{\varphi}^2 \quad : \text{evaluate when } \varphi \text{ is known}$$

same eq as pendulum:

$$ml\ddot{\varphi} = -mg\sin\varphi$$

$$\ddot{\varphi} = -\frac{g}{l}\sin\varphi$$

When $|\varphi| \ll 1$:

$$\ddot{\varphi} = -\frac{g}{R}\varphi \qquad \varphi = A\cos\omega t + B\sin\omega t \qquad \omega = \sqrt{\frac{g}{R}}$$

Otherwise: "elliptic integral".

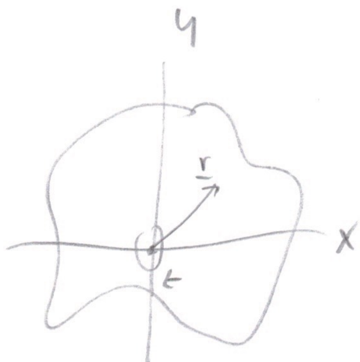
$\Rightarrow$

Object rotating about a fixed axis $(\hat{z})$

w = angular velocity about $\hat{z}$

$= \frac{d\varphi}{dt}$ $\quad \varphi$ = plane polar angle

In any plane z = constant:



$\underline{r} = (r\cos\varphi, r\sin\varphi)$

$\xrightarrow[\theta = wt]{\text{rotate by}} (r\cos(\varphi+\theta), r\sin(\varphi+\theta))$

$\equiv \underline{r}'$

$x' = r\cos\varphi\cos\theta - r\sin\varphi\sin\theta = x\cos\theta - y\sin\theta$

$y' = r\sin\varphi\cos\theta + r\cos\varphi\sin\theta = x\sin\theta + y\cos\theta$

$z' = z$

or $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \boxed{\underline{r}' = \underline{\underline{R}}_z(\theta) \cdot \underline{r}}$

Infinitesimal version $|d\theta| \ll 1$: $\quad x' \hat{=} x - y\,d\theta$

$\quad y' \hat{=} y + x\,d\theta \quad + O(d\theta)^2$

& since $d\theta = w\,dt$ $\quad \boxed{\begin{aligned} dx &= -y\,w\,dt \\ dy &= x\,w\,dt \end{aligned}}$

Turn this into a general rule:

let $\underline{\omega} = \omega\hat{z} = \omega\hat{\omega}$ angular velocity vector

rotation axis

notice $\underline{\omega}\,dt \times \underline{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega dt \\ x & y & z \end{vmatrix} = \hat{x}(-\omega dt\, y) + \hat{y}(\omega dt\, x)$

$= d\underline{r}$!

so $\dfrac{d\underline{r}}{dt} = \underline{\omega} \times \underline{r}$

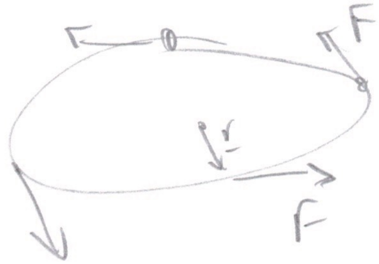
Here $\underline{r}$ = vector from origin to some body point

but the same thing would apply to any vector $\underline{\xi}$ fixed in the body: $\underline{\xi} = \underline{r}_1 - \underline{r}_2$ and $\underline{r}_{1,2}$ change this way, & the eq. is linear.

So $\boxed{\dfrac{d\underline{\xi}}{dt} = \underline{\omega} \times \underline{\xi}}$ for any vector $\underline{\xi}$ fixed in the rotating object.

Motivation for torque + angular momentum:

tangential force needed to rotate something in a circle



same result at different $\underline{r}$ where $\hat{F}$ is different

common feature:

$\underline{r} \times \underline{F}$ has same magnitude + direction $\parallel$ axis

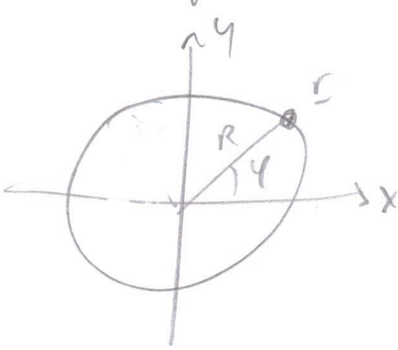
$\rightarrow$ define torque $\underline{\Gamma} = \underline{r} \times \underline{f}$ to characterize driving force

Notice $\frac{d}{dt}(\underline{r} \times \underline{p}) = \dot{\underline{r}} \times \underline{p} + \underline{r} \times \dot{\underline{p}} \rightarrow \underline{r} \times \underline{F}$

↳ interesting quantity

$\underline{l} = \underline{r} \times \underline{p}$ = angular momentum

Example: particle moving in a vertical circle



$$\dot{\underline{r}} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi} \rightarrow R\dot{\varphi}\hat{\varphi}$$

$$\underline{l} = \underline{r} \times \underline{p} = (R\hat{r}) \times (mR\dot{\varphi}\hat{\varphi})$$

$$= mR^2\dot{\varphi}\hat{z}$$

no torque: $\dot{\varphi}$ = constant

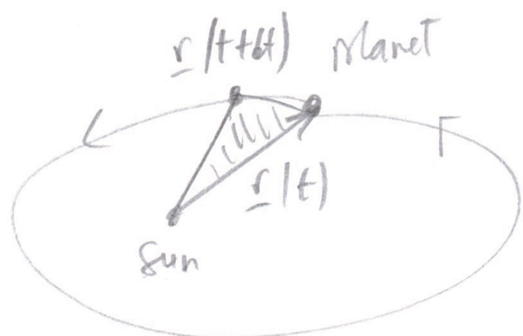
Apply "gravity" along $\hat{x}$: $\underline{f} = mg\,\hat{x}$

$$\underline{\Gamma} = \underline{r} \times \underline{f} = (R\hat{r}) \times (mg\hat{x}) = mgR(-\sin\varphi\,\hat{z})$$

$$= \frac{d\underline{l}}{dt} = mR^2\ddot{\varphi}\,\hat{z}$$

$$\rightarrow \ddot{\varphi} = -\frac{g}{R}\sin\varphi \quad \text{again}$$

Example 2 - planets



$$\underline{r}(t+dt) = \underline{r}(t) + \underline{v}(t)\,dt$$

$$= \underline{r}(t) + \underline{p}(t)/m \; dt$$

$$\text{Shaded area} = \frac{1}{2}|\underline{r} \times d\underline{r}|$$

$$= \frac{1}{2m}|\underline{r} \times \underline{p}|\,dt = \frac{l}{2m}\,dt$$

So $\dfrac{dA}{dt} = \dfrac{d}{dt}$(area swept out by orbiting planet)

$$= \frac{l}{2m}$$

Here $\underline{F}$ = gravity $\propto -\hat{r}$ so $\underline{\Gamma} = \underline{r} \times \underline{F} = 0$

"central force"

$$\rightarrow \frac{dA}{dt} = \text{constant}$$

"Kepler's Second law"

N-particle case: $\underline{L} \equiv \sum_{i=1}^{N} \underline{l}_i = \sum \underline{r}_i \times \underline{p}_i$

$$\dot{\underline{L}} = \sum_i \underline{r}_i \times \underline{F}_i \quad \text{as above.}$$

$$= \sum_i \underline{r}_i \times \Big(\underset{\uparrow \text{ gravity, E\&M}}{\underline{F}_i^{ext}} + \sum_{j \neq i} \underline{F}_{ij} \Big)$$

$\uparrow$ force on i from j

$$\equiv \underbrace{\sum_i \underline{r}_i \times \underline{F}_i^{ext}}_{\underline{\Gamma}^{ext} = \text{external torque}}$$

$$\sum_{i \neq j} \underline{r}_i \times \underline{F}_{ij} = \sum_{j \neq i} \underline{r}_j \times \underline{F}_{ji} = -\sum_{i \neq j} \underline{r}_j \times \underline{F}_{ij}$$

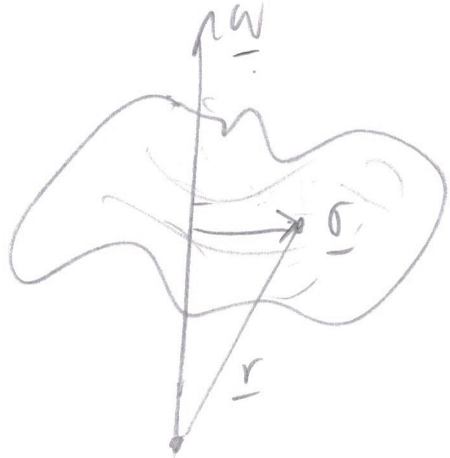
$$= \underline{\Gamma}^{ext} + \underbrace{\frac{1}{2} \sum_{i \neq j} (\underline{r}_i - \underline{r}_j) \times \underline{F}_{ij}}_{= 0 \text{ for central forces}}$$

$\therefore$ $\boxed{\dot{\underline{L}} = \underline{\Gamma}^{ext}}$ when internal forces are central.

This looks like $\dot{\underline{P}} = \underline{F}^{ext}$; need analog of $\underline{P} = M\underline{V}$

$\rightarrow$ $\underline{L} = I \underline{\omega}$??

Dynamics for a fixed rotation axis



object rotating at ω about $\hat{\omega}$

write $\underline{r} = \underline{\sigma} + \alpha \hat{\omega}$

$\underline{\sigma}$: part in plane $\perp \hat{\omega}$; $\alpha \hat{\omega}$: part along $\hat{\omega}$

so $\underline{v} = \underline{\omega} \times \underline{r} = \underline{\omega} \times \underline{\sigma} + \cancel{\underline{\omega} \times (\alpha \hat{\omega})} \to 0$

$$\underline{l} = \underline{r} \times (m \underline{v}) = (\underline{\sigma} + \alpha \hat{\omega}) \times (m \underline{\omega} \times \underline{\sigma})$$

$$= m \Big[\underbrace{\underline{\sigma} \times (\underline{\omega} \times \underline{\sigma})}_{m\sigma^2 \underline{\omega}} + \underbrace{\alpha \hat{\omega} \times (\underline{\omega} \times \underline{\sigma})}_{\text{in the plane } \perp \underline{\omega}} \Big]$$

so $l_\omega = \underline{l} \cdot \hat{\omega} = \underline{l}$ along rotation axis $= m\sigma^2 \omega$

This argument applies to all mass points in the object

so $\hat{\omega} \cdot \underline{L} = L_\omega = \sum_i m_i \sigma_i^2 \omega = I \omega$

σ_i: distance from axis; I: moment of inertial

$$\frac{d}{dt} L_\omega = \frac{d}{dt}(\hat{\omega} \cdot \underline{L}) = \hat{\omega} \cdot \frac{d\underline{L}}{dt} = \hat{\omega} \cdot \underline{\Gamma}^{ext}$$

$$\Rightarrow \boxed{I \dot{\omega} = \Gamma_\omega} = \text{torque about rotation axis}$$