Linear momentum

1 particle: define $\underline{p} = m\underline{v}$ so $\frac{d\underline{p}}{dt} = \underline{F}$

just a definition, rescaling of velocity

But: useful for multi-particle situations where different parts of the system move differently

connection to physical symmetries

connection to QM - particles have wave functions $\psi(\underline{r},t)$ which distort with no unique speed but $p_x \to -i\hbar \frac{\partial}{\partial x}$ etc. is well-defined

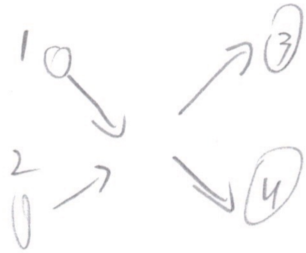
Hamiltonian formalism uses $\underline{r}$ and $\underline{p}$ $\Rightarrow$ QM

Many particles: define $\underline{P} = \sum_{i=1}^{N} m_i \underline{v}_i = M \underline{V}_{CM}$

so $\frac{d\underline{P}}{dt} = \sum_i m_i \dot{\underline{v}}_i = M \frac{d^2\underline{R}}{dt^2} = \underline{F}^{ext}$

Utility:

(1) Collisions



individual $\underline{v}_i$ change but $\underline{F}^{ext} = 0 \Rightarrow \underline{P} = \text{constant}$

useful for analyzing final states

e.g. if $(1) + (2) \to (3)$ then $\underline{p}_{\text{before}} = \underline{p}_{\text{after}}$

$$\underline{p}_1 + \underline{p}_2 = \underline{p}_3 = m_3 \underline{v}_3$$

so v of product known

Otherwise if $(A) \to (B) + (C)$

then in rest frame of A $\underline{p}_A = 0 = \underline{p}_B + \underline{p}_C$

so $m_B \underline{v}_B = - m_C \underline{v}_C$ or $\hat{v}_B = - \hat{v}_C$

$$v_B / v_C = m_C / m_B$$

For ≥ 4 particles not that useful,

need to consider cons. of energy also - next chapter

(2) when $\underline{F}^{\text{ext}} = 0$ system is not directed to move in any particular direction

$\to$ all points in space are equivalent

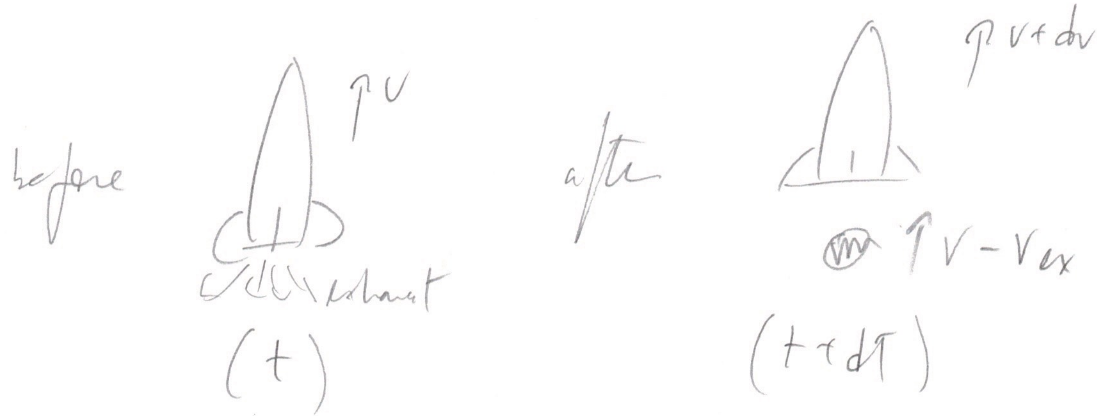
$\to$ "translation invariance" $\to$ $\underline{p}$ = const

Example of a general relation btw symmetry + existence of a conserved quantity : more later

(3) If mass added/removed, $\underline{p}$ is what responds to $\underline{F}$

$\to$ eq of motion

Rockets



$$p(t) = m(t)\, v(t)$$

$$p(t+dt) = (\underset{dm<0}{m+dm})(\underset{dv>0}{v+dv}) + (\underset{\text{pos.}}{-dm})(v-v_{ex})$$

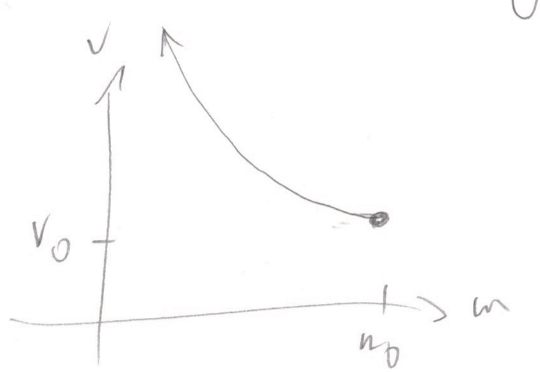
no gravity

$$= \underset{p(t)}{mv} + m\,dv + \cancel{v\,dm} + \cancel{dm\,dv} - \cancel{v\,dm} + v_{ex}\,dm$$

so $\quad m\,dv = -v_{ex}\,dm$

$$\frac{v}{v_{ex}} = -\log m + \underset{\frac{v_0}{v_{ex}} + \log m_0}{\text{const}}$$

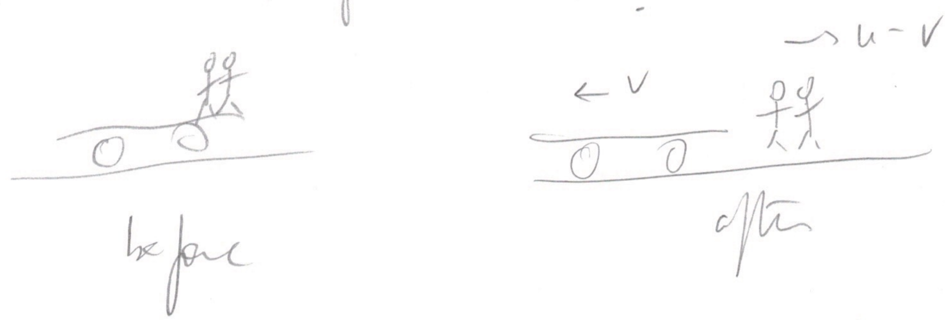
or $\quad v = v_0 + v_{ex} \log \frac{m_0}{m}$



Gravity?
example in PS 2

Example (3.4) Two hobos (m_h) on an RR car (m_c), jump off with relative velocity u. No friction.

(a) Both jump off one end simultaneously, car recoils with speed $v = ?$



$$p = 0 = -m_c v + 2m_h(u-v) \qquad v = \frac{2m_h u}{m_c + 2m_h}$$

(b) One jumps, then the 2nd:

1st jump: $0 = -(m_c+m_h)v_1 + m_h(u-v_1)$

$$\rightarrow \quad v_1 = \frac{m_h}{m_c+2m_h}\, u$$

2nd:

$$-(m_c+m_h)v_1 = -m_c v_2 + m_h(u-v_2)$$

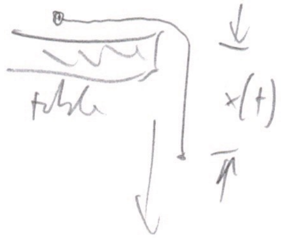
$\vdots$

$$v_2 = \frac{3m_h + 2m_c}{2m_h + 2m_c} \cdot \frac{2m_h}{2m_h + m_c}\, u = \frac{3m_h + 2m_c}{2m_h + 2m_c}\, v^{(a)}$$

$$> v_1 \text{ or } v^{(a)}$$

The hobo who jumped first is irrelevant here.

Example: uniform chain of mass m & length l sits on a table, with a segment x_0 hanging over the edge. Gravity will pull the chain off (no friction) - motion?



$\downarrow g$

weight of hanging portion $m\frac{x(t)}{l}g$
but chain moves as a unit

So $\frac{dp}{dt} = m\ddot{x} = m\frac{x}{l}g$ or $\ddot{x} = \alpha^2 x$ $\alpha = \sqrt{\frac{g}{l}}$

$$\rightarrow \quad x(t) = A e^{\alpha t} + B e^{-\alpha t}$$

$$x(0) = x_0 \rightarrow A + B = x_0$$

$$\dot{x}(0) = 0 \rightarrow \alpha(A - B) = 0 \rightarrow A = B = \frac{x_0}{2}$$

$$x(t) = x_0 \cosh \alpha t$$

Chain comes off when $x(t) = l$ or $t = \frac{1}{\alpha}\cosh^{-1}\frac{l}{x_0}$

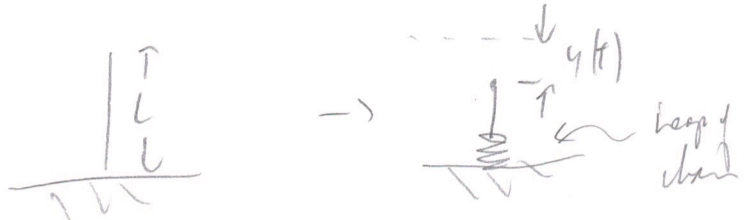
at which point $v = \alpha x_0 \sinh \alpha t = \alpha x_0 \sinh\left(\cosh^{-1}\frac{l}{x_0}\right)$.

To look at this suppose $l \gg x_0$ & use $\cosh z \simeq \sinh z = \frac{1}{2}e^z$

so $v \rightarrow \alpha x_0 \cdot \frac{l}{x_0} = \sqrt{gl}$

If the chain had fallen freely through a distance l, $v = \sqrt{2gl}$
which is larger: less v because less effective g

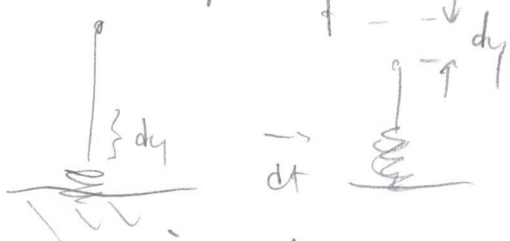
A uniform flexible frictionless chain falls vertically on a table; what force does it exert?



$y(t)$ = distance top falls
= length on the table

$F \neq$ weight on table because falling part transfers momentum.

Look at a segment of length dy before & after impact:



momentum transferred $= -\lambda dy \cdot v = dp$

$$F = \text{weight} + \frac{dp}{dt} = -\lambda g y + \frac{(-\lambda dy \cdot v)}{dt} = -\lambda g y - \lambda v^2$$

but for free fall $v^2 = 2gy$

$$\rightarrow F = -3\lambda g y$$

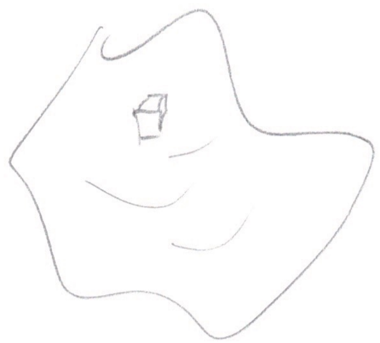
Once the whole chain is on the table $F \rightarrow -\lambda g L$!

difference: moving chain is transferring momentum to the table, more & more as the chain speeds up, but at rest there's just gravity.

Continuous systems –

break the continuum into regions ΔV centred at $\mathbf{r}_i$

with $m_i = \rho(\mathbf{r}_i)\,\Delta V \qquad \rho = \text{density}$



$$\mathbf{R} = \frac{1}{\sum m_i} \sum m_i \mathbf{r}_i$$

$$\longrightarrow \frac{1}{\sum_i \rho(\mathbf{r}_i)\Delta V} \sum_i \rho(\mathbf{r}_i)\Delta V\,\mathbf{r}_i$$

$$\xrightarrow[\Delta V \to 0]{} \frac{1}{\int \rho(\mathbf{r})\,dV} \int \rho(\mathbf{r})\,\mathbf{r}\,dV$$

$$= \frac{1}{M} \int \rho(\mathbf{r})\,\mathbf{r}\,dV$$

then $\mathbf{P} = \frac{1}{M} \int \rho(\mathbf{r})\,\dot{\mathbf{r}}\,dV$ etc.

To go back to the continuum:

a point particle at $\mathbf{r}_i$ has density $m_i\,\delta(\mathbf{r}-\mathbf{r}_i)$

so $\rho(\mathbf{r}) = \sum_i m_i\,\delta(\mathbf{r}-\mathbf{r}_i)$ for a set of point particles.

Here "$\delta(\mathbf{r})$" $= \delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z)$

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{2\pi\varepsilon^2}}\, e^{-x^2/2\varepsilon^2}$$