Coupled Oscillations in General:

look at equilibria of a Lagrangian system with n degrees of freedom

$L = T - V$ with gen'l coordinates $\{q_1 \cdots q_n\}$

If there is an equilibrium at $\underline{q} = \underline{q}_0 = \{q_{10} \cdots q_{n0}\}$

then $\left.\dfrac{\partial V}{\partial q_i}\right|_{\underline{q}_0} = 0$ for all $i = 1, \cdots, n$

For stability, write $q_i = q_{i0} + x_i$ with $|x_i|$ small,

expand
$$V(\underline{q}) = \underbrace{V(\underline{q}_0)}_{\substack{\text{constant}\\ \text{equil PE}}} + \sum_i \underbrace{\left.\frac{\partial V}{\partial q_i}\right|_{\underline{q}_0}}_{0} x_i + \frac{1}{2}\underbrace{\sum_{ij} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\underline{q}_0} x_i x_j}_{\text{quadratic}} + \underbrace{\cdots}_{\substack{\text{higher}\\ \text{order}}}$$

An equilibrium is stable if it costs energy to deviate from it,

or (quadratic) ≥ 0

or $K_{ij} = \left.\dfrac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\underline{q}_0}$ = positive matrix

$\sum_{ij} K_{ij} x_i x_j > 0$ for any $\underline{x} \neq 0$

[$\sum K_{ij} x_i x_j > 0$ = positive definite, too strong a condition since "zero-modes" occur.]

For dynamics, look at T also:

$$T = \sum_a \frac{1}{2} m_a \dot{\underline{r}}_a^2 \xrightarrow[\underline{r}_a = \underline{r}_a(\underline{q})]{} \sum_{ij} \frac{1}{2} \underbrace{\left(\sum_a \frac{\partial \underline{r}_a}{\partial q_i} \cdot \frac{\partial \underline{r}_a}{\partial q_j} \right)}_{\text{positive definite } (m_{ij})} \dot{q}_i \dot{q}_j$$

Expand T about q_0 also:

$$T \to \frac{1}{2} \sum_{ij} \underbrace{m_{ij}(\underline{q}_0 + \underline{x})}_{= \underline{m}_{ij}(\underline{q}_0) + O(\underline{x})} \dot{x}_i \dot{x}_j$$

So $L = \sum_{ij} \frac{1}{2} M_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} \sum_{ij} K_{ij} x_i x_j + O(x^3)$

$= \frac{1}{2} \dot{\underline{x}}^T \underline{\underline{M}}\, \underline{x} - \frac{1}{2} \underline{x}^T \underline{\underline{K}}\, \underline{x}$ $\quad \underline{\underline{M}}, \underline{\underline{K}}$ = constant matrices

Eq of motion $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \to \sum_j M_{ij} \ddot{x}_j = -\sum_j K_{ij} x_j$

Look for normal mode solutions $x_j(t) = a_j e^{-i\omega t}$

$\to \sum_j (K_{ij} - \omega^2 M_{ij}) a_j = 0$ or $(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) \cdot \underline{a} = 0$

$\to$ Variant eigenvalue problem because $\underline{\underline{M}} \neq m \underline{\underline{1}}$

Properties of eigenvalues $\lambda \equiv \omega^2$ and eigenvectors:

$\underline{\underline{M}}$ & $\underline{\underline{K}}$ are real + symmetric $\to$ Hermitian matrices

$\underline{\underline{K}} \cdot \underline{a} = \lambda \underline{\underline{M}} \cdot \underline{a} \to \underbrace{\underline{a}^\dagger \cdot \underline{\underline{K}} \cdot \underline{a}} = \lambda\, \underline{a}^\dagger \underline{\underline{M}}\, \underline{a}$

$= (\underline{\underline{K}} \cdot \underline{a})^\dagger \cdot \underline{a} = (\lambda \underline{\underline{M}} \cdot \underline{a})^\dagger \cdot \underline{a} = \lambda^* \underline{a}^\dagger \underline{\underline{M}}\, \underline{a}$

so $\lambda = \lambda^*$

and $\underline{\underline{K}} \underline{a}^* = \lambda \underline{\underline{M}}\, \underline{a}^*$ $\to$ $\underline{a}$ and $\underline{a}^*$ are eigenvectors

$\to \frac{1}{2}(\underline{a} + \underline{a}^*)$ = real vector = eigenvector so $\lambda = \dfrac{\underline{a}^T \cdot \underline{\underline{K}} \cdot \underline{a}}{\underline{a}^T \cdot \underline{\underline{M}} \cdot \underline{a}} = \dfrac{\text{pos or zero}}{\text{pos.}}$ = positive or zero

Also, if λ & λ' are distinct

$$\underline{a}'^T \cdot \underline{\underline{K}} \cdot \underline{a} = \underline{a}'^T \cdot \lambda \underline{\underline{M}} \cdot \underline{a} = \lambda\, \underline{a}'^T \cdot \underline{\underline{M}} \cdot \underline{a}$$

$$\hookrightarrow = (\underline{\underline{K}} a')^T \cdot a = (\lambda' \underline{\underline{M}} a') \cdot \underline{a} = \lambda' \underline{a}'^T \cdot \underline{\underline{M}} \cdot \underline{a}$$

so $\underline{a}'^T \cdot \underline{\underline{M}} \cdot \underline{a} = 0$; orthogonality with "weight" $\underline{\underline{M}}$

Therefore: eigenvalues & eigenvectors are real

if all eigenvalues distinct get n "orthogonal" eigenvectors $\underline{a}_i^T \underline{\underline{M}}\, \underline{a}_j = \begin{cases} 0 & i \neq j \\ \text{positive} & i = j \end{cases}$

normalization: $\underline{a}_i^T \underline{\underline{M}}\, \underline{a}_j = \delta_{ij}$

$$\rightarrow \underline{a}_i^T \cdot \underline{\underline{K}} \cdot \underline{a}_j = \lambda_i \delta_{ij}.$$

If $\{\lambda_i\}$ not distinct, use Gram-Schmidt procedure with weight $\underline{\underline{M}}$:

if $\underline{\underline{M}}\, \underline{a}_{1,2} = \lambda \underline{\underline{K}}\, a_{1,2}$ then eigenfunctions are a_1 and $\hat{a}_2 = N\left(a_2 - a_1 \left(\underline{a}_2^T \underline{\underline{M}}\, \underline{a}_1\right)\right)$

Then: let $\underline{\underline{A}} = (\underline{a}_1, \underline{a}_2 \cdots \underline{a}_n)$ = n×n matrix whose columns are $\{\underline{a}_i\}$

$$\underline{\underline{A}}^T \underline{\underline{M}}\,\underline{\underline{A}} = \begin{pmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_n^T \end{pmatrix} \cdot \left(\underline{\underline{M}}\underline{a}_1, \underline{\underline{M}}\underline{a}_2 \dots \underline{\underline{M}}\underline{a}_n \right) = \left(\underline{a}_i^T \underline{\underline{M}}\, \underline{a}_j \right) = \underline{\underline{1}}$$

$$\underline{\underline{A}}^T \underline{\underline{K}}\,\underline{\underline{A}} = \begin{pmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_n^T \end{pmatrix} \cdot \left(\lambda_1 \underline{\underline{K}}\underline{a}_1, \lambda_2 \underline{\underline{K}}\underline{a}_2 \dots \right) = \left(\delta_{ij} \lambda_i \right)$$

$$= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \equiv \underline{\underline{\Lambda}}$$

General solution of the original problem is

$$\underline{x}(t) = \underbrace{\mathrm{Re}}_{\text{real part}} \sum_k \underbrace{c_k}_{\text{constants}} \underline{a}_k e^{-i\omega_k t} = \text{linear combination of eigenvectors}$$

$$\text{or } x_i(t) = \mathrm{Re} \sum_k c_k \underbrace{(\underline{a}_k)_i}_{= A_{ik}} e^{-i\omega_k t}$$

$$= \mathrm{Re}\, \underline{\underline{A}} \cdot \underline{c}(t), \quad \underline{c}(t) = \begin{pmatrix} c_1 e^{-i\omega_1 t} \\ \vdots \\ c_n e^{-i\omega_n t} \end{pmatrix}$$

I.C.: $\underline{x}(0) = \mathrm{Re}\left(\underline{\underline{A}} \cdot \underline{c}(0) \right)$
$\dot{\underline{x}}(0) = -\mathrm{Im}\left(\underline{\underline{A}} \cdot \underline{\underline{\Lambda}}^{1/2} \cdot \underline{c}(0) \right)$ $\Rightarrow \mathrm{Re} + \mathrm{Im}\; c_i$

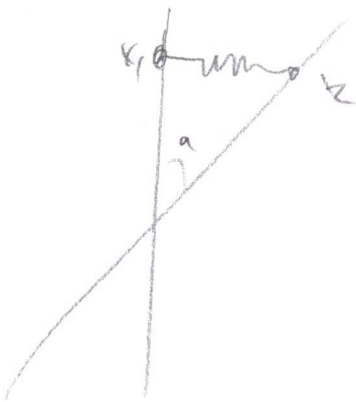
Normal modes: let $\underline{x} = \underline{\underline{A}} \cdot \underline{y}$

$$L = \frac{1}{2} \dot{\underline{x}}^T \underline{\underline{M}}\, \dot{\underline{x}} - \frac{1}{2} \underline{x}^T \underline{\underline{K}}\, \underline{x}$$

$$= \frac{1}{2} \dot{y}^T \underbrace{(A^T M A)}_{= \underline{\underline{1}}} \dot{y} - \frac{1}{2} y^T \underbrace{(A^T K A)}_{= \underline{\underline{\Lambda}}} y$$

$$= \sum_i \frac{1}{2} \left(\dot{y}_i^2 - \omega_i^2 y_i \right) = \text{uncoupled SHO's}$$

Initial example again



$$T = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right)$$

$$V = + \frac{1}{2} k \left(x_1^2 + x_2^2 - 2 x_1 x_2 \cos\alpha \right)$$

$$\underline{\underline{M}} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad \underline{\underline{V}} = \begin{pmatrix} k & -k\cos\alpha \\ -k\cos\alpha & k \end{pmatrix}$$

$$0 = \det \left| \omega^2 \underline{\underline{M}} - \underline{\underline{V}} \right| = \det \begin{pmatrix} m\omega^2 - k & +k\cos\alpha \\ +k\cos\alpha & m\omega^2 - k \end{pmatrix}$$

$$= \left(m\omega^2 - k \right)^2 - k^2 \cos^2\alpha$$

$$= \left(m\omega^2 \right)^2 - 2k\, m\omega^2 + k^2 \left(1 - \cos^2\alpha \right)$$

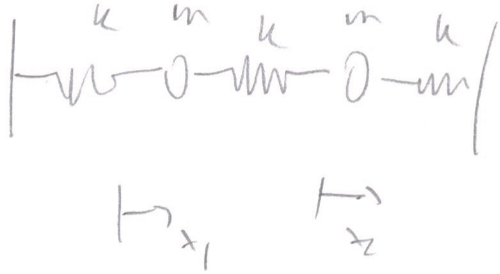
$$= \left(m\omega^2 - k(1+\cos\alpha) \right) \left(m\omega^2 - k(1-\cos\alpha) \right)$$

$$\begin{pmatrix} k\cos\alpha & k\cos\alpha \\ k\cos\alpha & k\cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad \begin{pmatrix} -k\cos\alpha & k\cos\alpha \\ k\cos\alpha & -k\cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$x = -y$ $\qquad$ $x = +y$

$$\underline{\xi}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \underline{\xi}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

example from Taylor (largely as initial aspect here)



x_i = displacement of masses from equil

$$T = \tfrac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) \qquad V = \frac{k}{2}\left[x_1^2 + (x_1 - x_2)^2 + x_2^2\right]$$

$$= \frac{k}{2}\left[2x_1^2 + 2x_2^2 - 2x_1 x_2\right]$$

$$\underline{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad \underline{V} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$$

$$\det(\omega^2 \underline{M} - \underline{V}) = \det\begin{pmatrix} m\omega^2 - 2k & k \\ k & m\omega^2 - 2k \end{pmatrix}$$

$$= (m\omega^2 - k)(m\omega^2 - 3k)$$

$$\begin{pmatrix} -k & k \\ k & -k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$x = y$

no middle stretch

in phase

force from separate springs at the ends

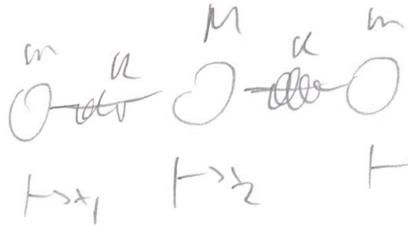
$$\begin{pmatrix} k & k \\ k & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$x = -y$

middle has double stretch

effective $k = k + 2k = 3k$

Ex 4 Triatomic molecule (11.32) in 1-d oscillation



x_i = displacements from unstretched springs

$$L = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_3^2\right) + \frac{M}{2}\left(\dot{x}_2^2\right) - \frac{1}{2}k(x_1 - x_2)^2 - \frac{1}{2}k(x_2 - x_3)^2$$

$$\underline{\underline{M}} = \begin{pmatrix} m & & 0 \\ & M & \\ 0 & & m \end{pmatrix} \qquad \underline{\underline{K}} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

$$\det\left(\omega^2 \underline{\underline{M}} - \underline{\underline{K}}\right) = \det\begin{pmatrix} \omega^2 m - k & +k & 0 \\ +k & \omega^2 M - 2k & +k \\ 0 & +k & \omega^2 m - k \end{pmatrix}$$

$$= (\omega^2 m - k)\left[(\omega^2 M - 2k)(\omega^2 m - k) - k^2\right]$$

$$- k\left[+k(\omega^2 m - k)\right]$$

$$= (\omega^2 m - k)\left[\underbrace{\omega^4 mM - 2k\omega^2 m - k\omega^2 M}_{\omega^2(\omega^2 mM - 2km - kM)} + \cancel{2k^2} - \cancel{2k^2}\right]$$

so $\omega = 0$ is an eigenvalue; eigenvector

$$\begin{pmatrix} -k & +k & 0 \\ -k & -2k & +k \\ 0 & +k & -k \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad : x = y = z$$

$$\hat{a}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

* uniform translation of the whole molecule, no vibration
"zero-mode" - neutrally stable motion, no cost
in energy

Other modes $m\omega^2 = k$ $\omega = \omega_0$

$$\begin{pmatrix} 0 & k & 0 \\ k & k\left(\frac{M}{m}-2\right) & k \\ 0 & k & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \rightarrow \quad \begin{matrix} y = 0 \\ x = -z \end{matrix}$$

$$\hat{a}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$! outer atoms move back & forth out of phase while middle atom sits

& $\omega^2 = k\left(\frac{2}{M} + \frac{1}{m}\right) = \omega_0^2\left(2\frac{m}{M}+1\right)$

... $\hat{a}_3 \propto \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$ outer atoms in phase with same amplitude, mean out of phase w diff ampl.

→ Notice $\hat{a}_i^T \underline{\underline{M}}\, \hat{a}_j = \delta_{ij}$!

Better(?) method: to eliminate translation mode fix the CM

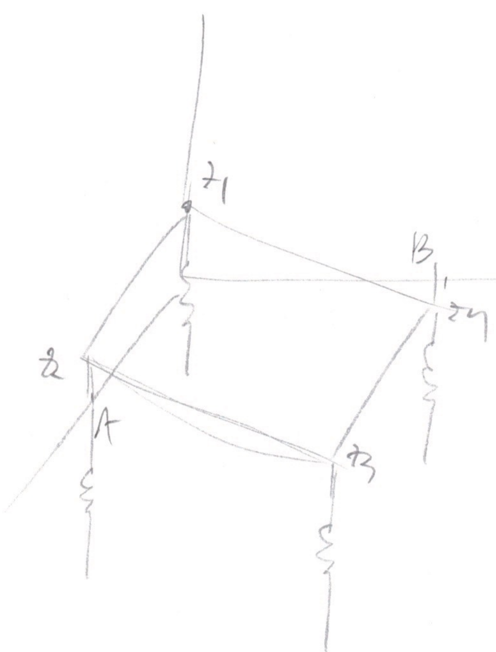
$m x_1 + M x_2 + m x_3 = 0$ say $\rightarrow x_2 = -\frac{m}{M}(x_1 + x_3)$

$$\left.\begin{aligned} T &\rightarrow \tfrac{1}{2} m\left(\dot{x}_1^2 + \dot{x}_3^2\right) + \frac{m^2}{2M}\left(\dot{x}_1 + \dot{x}_3\right)^2 \\ V &\rightarrow \tfrac{1}{2} k\left(x_1 + \frac{m}{M}(x_1 + x_3)\right)^2 + 1 \rightarrow 2 \end{aligned}\right\} \quad 2\times 2 \text{ system} \ldots$$

Degeneracy example: vibrating rigid bed

4 masses m at corners of rectangular $(A \times B)$ massless, rigid frame

Held on <u>vertical</u> springs (k)



$$L = \sum_1^4 \frac{m}{2} \dot{z}_i^2 - \sum \frac{k}{2} z_i^2$$

But rigidity constrains z_i:

midpoint of 1-3 diagonal at $\frac{z_1 + z_3}{2}$ ⎫ midpoints of diagonals

" 2-4 " $\frac{z_2 + z_4}{2}$ ⎭ coincide if rigid

$$\rightarrow \quad z_1 + z_3 = z_2 + z_4$$

Alternate approach:

$$(\underline{r}_2 - \underline{r}_1) \times (\underline{r}_4 - \underline{r}_1) = \text{normal to 1-3 corner}$$

$$= (\underline{r}_4 - \underline{r}_3) \times (\underline{r}_2 - \underline{r}_3) = \quad " \quad \text{2-4 corner}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A & 0 & z_2 - z_1 \\ 0 & B & z_4 - z_1 \end{vmatrix} = -\hat{x} B(z_2 - z_1) - \hat{y} A(z_4 - z_1) + AB\hat{z}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -A & 0 & z_4 - z_3 \\ 0 & -B & z_2 - z_3 \end{vmatrix} = \hat{x} B(z_4 - z_3) + \hat{y} A(z_2 - z_3) + AB\hat{z}$$

$\to -(z_2 - z_1) = z_4 - z_3$, $-(z_4 - z_1) = z_2 - z_3$: symmetry

$$L \to \frac{1}{2} m \left(\dot z_1^2 + \dot z_2^2 + \dot z_3^2 + (\dot z_1 + \dot z_3 - \dot z_2)^2 \right)$$
$$- \frac{k}{2} \left(z_1^2 + z_2^2 + z_3^2 + (z_1 + z_3 - z_2)^2 \right)$$

$$= \frac{m}{2} \dot{\underline{z}}^T \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \dot{\underline{z}} - \frac{k}{2} \underline{z}^T \left(\text{same matrix} \right) \underline{z}$$

so $\quad \omega^2 \underline{\underline{M}} - \underline{\underline{K}} = (\omega^2 m - k) \otimes \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

$$\det(\;) = (\omega^2 m - k)^3 \det(\;''\;) = 4(\omega^2 m - k)^3$$

: triply-degenerate eigenvalue.

Because $\underline{\underline{M}} \propto \underline{\underline{K}}$, any vector is an eigenvector,

Use physical reasoning & orthogonality $\hat a_i^T \underline{\underline{M}} \hat a_j = \delta_{ij}$

to fix them.

Obvious normal mode is $z_1 = z_2 = z_3 = z_4$: force is horizontal,

each spring sees stiffness k, $\omega^2 = k/m$, $\hat a_1 \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

To have $\hat a_1^T \cdot \underline{\underline{M}} \cdot \hat a_1 = 1$ note

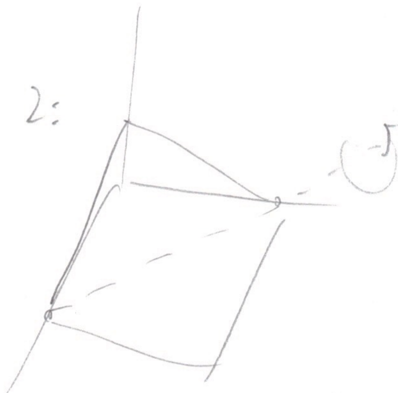
$$(1\,1\,1) \underline{\underline{M}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \dots = 2m, \quad \hat a_1 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

note $\underline{\underline{M}}\,\hat{a}_1 = \sqrt{\frac{2}{m}}\begin{pmatrix}1\\0\\1\end{pmatrix}$.

Could choose $\underline{a}_2 \propto \begin{pmatrix}1\\0\\-1\end{pmatrix} \rightarrow \hat{a}_2 = \frac{1}{\sqrt{2m}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$

which has $z_1 = -z_3 = 1,\ z_2 = z_4 = 0$

2:



rotate about 2-4 axis

3:

rotate about 1-3 axis

or $\underline{a}_3 \propto \begin{pmatrix}0\\1\\0\end{pmatrix} \rightarrow a_3 = \frac{1}{\sqrt{2m}}\begin{pmatrix}0\\1\\0\end{pmatrix}$

with $z_2 = -z_4 = 1,\ z_1 = z_3 = 0$