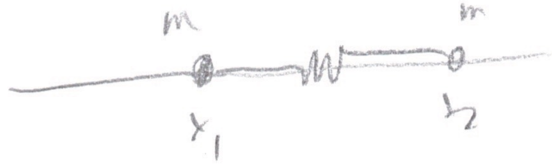
Simple Intro to coupled oscillations



2 masses on a line, connected by spring

$$L = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}k(x_1 - x_2)^2$$

let $x_c = \frac{1}{2}(x_1 + x_2)$ $\quad r = x_1 - x_2$ $\rightarrow$ $x_1 = x_c + r/2$, $\quad x_2 = x_c - r/2$

$$L \rightarrow m\left(\dot{x}_c^2 + \frac{1}{2}\dot{r}^2\right) - \frac{1}{2}k r^2$$

$\frac{d}{dt}\dot{x}_c = 0$ $\qquad x_c = \alpha + \beta t$

$\frac{d}{dt}\frac{m}{2}\dot{r} = -kr$ $\qquad r =$ Harmonic Oscillator $\left(\frac{m}{2}, k\right)$ $\quad \omega = \sqrt{\frac{2k}{m}}$

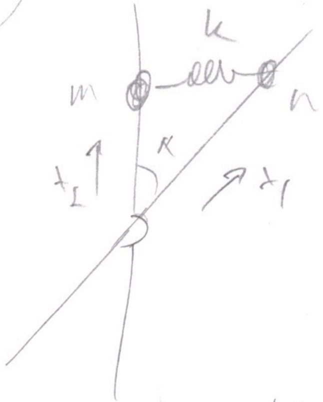
∴ 2 "modes" added together

x_c translates freely ↔ HO with $\omega = 0$

x_r oscillates with high ω (lower eff mass)

These are decoupled except for IC → normal modes.

Ex 1 Scissors:



2 rods, frictionless, mass m sliding on each; held near-touching at α, spring (k) btw masses. Motion?

If either mass is displaced from crossing point have P.E. = E > 0 & restoring force, so system will oscillate somehow.

$$L = \frac{1}{2} m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2} k\Big(\underbrace{x_1^2 + x_2^2 - 2x_1 x_2 \cos\alpha}_{= |\underline{r}_1 - \underline{r}_2|^2}\Big)$$

$$m\ddot{x}_1 = -k\left(x_1 - x_2 \cos\alpha\right)$$
$$m\ddot{x}_2 = -k\left(x_2 - x_1 \cos\alpha\right)$$

[: $x_1 \pm x_2$ → alternate method →

Look for globally oscillating solution $x_1 = a_1 e^{-i\omega t}$, $x_2 = a_2 e^{-i\omega t}$

$$m\omega^2 a_1 = k(a_1 - a_2 \cos\alpha)$$
$$m\omega^2 a_2 = k(a_2 - a_1 \cos\alpha)$$

} 2 homogeneous lin eq for a_1,

or $$\begin{pmatrix} m\omega^2 - k & k\cos\alpha \\ k\cos\alpha & m\omega^2 - k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$ where $\underline{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-i\omega t}$

So $(m\omega^2 - k)^2 - k^2\cos^2\alpha = 0$

$$(m\omega^2)^2 - 2k(m\omega^2) + k^2\sin^2\alpha = 0$$

$$\omega_\pm^2 = \frac{1}{m}\left[k \pm \sqrt{k^2 - k^2\sin^2\alpha}\right] = \frac{k}{m}(1 \pm \cos\alpha)$$

Amplitudes:
$$\begin{pmatrix} \mp k\cos\alpha & k\cos\alpha \\ \pm k\cos\alpha & k\cos\alpha \end{pmatrix}\begin{pmatrix} a_{1\pm} \\ a_{2\pm} \end{pmatrix} = 0$$

so $a_{1+} = -a_{2+}$

$a_{1-} = +a_{2-}$ $\rightarrow$ $\underline{\xi}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\underline{\xi}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +1 \end{pmatrix}$

$\rightarrow$ These are 2 linear indep solutions of a 2nd order ODE so

solution is $\quad \underline{x} = A\underline{\xi}_+ e^{-i\omega_+ t} + B\underline{\xi}_- e^{-i\omega_- t}$

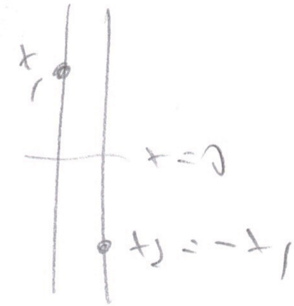
Really, A B complex $\rightarrow$ 4 real param

$\mathrm{Re}(\underline{x})$ = physical displacement has 4 constants

$\hookrightarrow$ $\underline{x}(0)$ + $\dot{\underline{x}}(0)$ $\rightarrow$

$\underline{x}_\pm$ are "normal modes" - specific osc. with both particles "in phase" (osc at same ω), and general solution = linear combination of normal modes limiting case $\alpha \to 0$

mode $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\omega^2 = \frac{2k}{m}$:

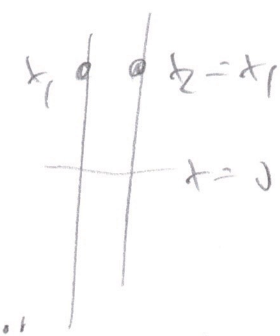


$$m\ddot{x}_1 = -k(x_1 - x_2) = -2kx_1$$

oscillate at $\omega^2 = \frac{2k}{m}$ about $x_1 = 0$

note $m\ddot{x}_2 = -k(x_2 - x_1)$

so $\frac{d^2}{dt^2} m(x_1 + x_2) = 0$ $\to$ center translates freely



mode $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\omega^2 = 0$

$$m\ddot{x}_1 = -k(x_1 - x_2) = 0$$

$\omega^2 = 0$ - free translation

NB: $\omega = 0 \leftrightarrow$ no force in that normal coordinate

$\leftrightarrow$ free translation

Nicer: rewrite in terms of $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$$L = \frac{1}{2}\,\underline{\dot{x}}^T \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \underline{\dot{x}} - \frac{1}{2}\,\underline{x}^T \begin{pmatrix} k & -k\cos\alpha \\ -k\cos\alpha & k \end{pmatrix} \underline{x}$$

$$= \frac{1}{2}\,\underline{\dot{x}}^T \underbrace{\underline{\underline{M}}}_{\text{mass matrix}} \underline{\dot{x}} - \frac{1}{2}\,\underline{x}^T \underbrace{\underline{\underline{V}}}_{\text{potential energy matrix}} \underline{x}$$

let $\underline{x} = \underline{\underline{A}}\,\underline{y}$ where $\underline{\underline{A}} = \left(\underline{\xi}_+, \underline{\xi}_-\right) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

notice $\underline{\underline{A}}^T \underline{\underline{A}} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \dots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

also $A^T M A = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$

and $A^T V A = \frac{k}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{\begin{pmatrix} 1+\cos\alpha & 1-\cos\alpha \\ -1-\cos\alpha & 1-\cos\alpha \end{pmatrix}}$

$$= \frac{k}{2} \begin{pmatrix} 2+2\cos\alpha & 0 \\ 0 & 2-2\cos\alpha \end{pmatrix} = \begin{pmatrix} m\omega_+^2 & 0 \\ 0 & m\omega_-^2 \end{pmatrix}$$

So
$$L = \frac{1}{2}\dot{\underline{y}}^T \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \dot{\underline{y}} - \frac{1}{2}\underline{y}^T \begin{pmatrix} m\omega_+^2 & 0 \\ 0 & m\omega_-^2 \end{pmatrix} \underline{y}$$

+ of $\underline{y} = \begin{pmatrix} y_+ \\ y_- \end{pmatrix}$

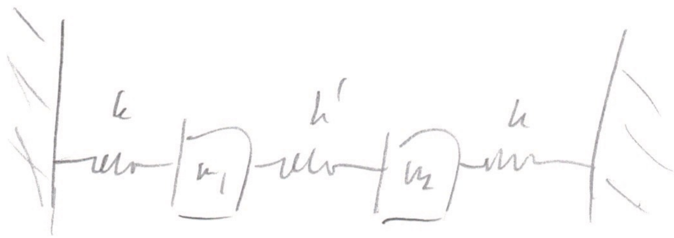
$$L \to \left(\frac{1}{2}m\dot{y}_+^2 - \frac{1}{2}m\omega_+^2 y_+^2\right) + \left(\frac{1}{2}m\dot{y}_-^2 - \frac{1}{2}m\omega_-^2 y_-^2\right)$$

+ $m\ddot{y}_i = -m\omega^2 y_i$

and its obvious that there are 2 independent oscillations with different frequency.

So : normal modes arise from diagonalizing the mass + PE matrices simultaneously.

Above "scissors problem" looks specific but exemplifies (= same algebra) as a few others:



1-d only
no friction

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{2}x_1^2 - \frac{k'}{2}(x_1 - x_2)^2 - \frac{k}{2}x_2^2$$

x_i = disp. from equilibrium position where all 3 springs are unextended.

$$\rightarrow \quad m\ddot{x}_1 = -kx_1 - k'(x_1 - x_2)$$
$$m\ddot{x}_2 = +k'(x_1 - x_2) - kx_2$$

Look for $x_i(t) = a_i e^{-i\omega t}$

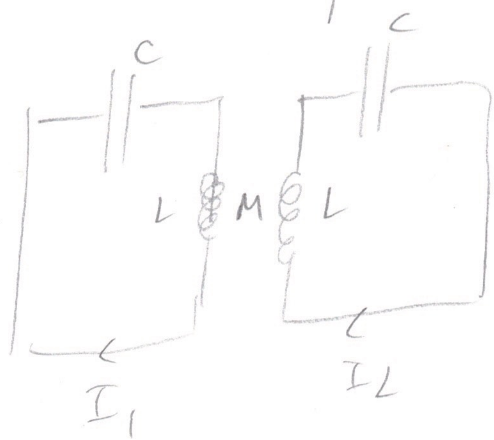
$$m\omega^2 a_1 = (k+k')a_1 - k'a_2$$
$$m\omega^2 a_2 = -k'a_1 + (k+k')a_2$$

Same problem with $k + k' \rightarrow k$

$k' \rightarrow k\cos\alpha$

More complicated (algebraically) if $m_1 \neq m_2$ or 3 different k's but similar ideas to analyze.

Electric Circuit Example



$$L\dot{I}_1 + \frac{Q_1}{C} + M\dot{I}_2 = 0$$

$$L\dot{I}_2 + \frac{Q_2}{C} + M\dot{I}_1 = 0$$

$\frac{d}{dt}$ (1st eq) : $\quad L\ddot{I}_1 + \frac{\dot{Q}_1}{C} + M\ddot{I}_2 = L\ddot{I}_1 + \frac{I_1}{C} + M\ddot{I}_2 \rightarrow$

$$= L\ddot{I}_1 + \frac{I_1}{C} + M\left(-\frac{I_2}{LC} - \frac{M\ddot{I}_1}{L}\right)$$

$$= \left(L - \frac{M^2}{L}\right)\ddot{I}_1 + I_1/C - \frac{M}{LC} I_2 = 0$$

likewise $\quad \left(L - \frac{M^2}{L}\right)\ddot{I}_2 + I_2/C - \frac{M}{LC} I_1 = 0$

Same form as 2-mass examples above.

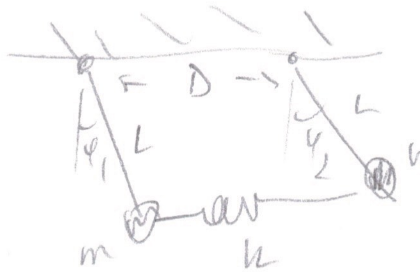
Look for $I_{1,2} \propto e^{-i\omega t} \cdot a_i$

$$-\omega^2\left(L - \frac{M^2}{L}\right)a_1 + a_1/C - \frac{M}{LC} a_2 = 0$$

2 2 1

$$\det\begin{pmatrix} \frac{1}{C} - \omega^2\left(L - \frac{M^2}{L}\right) & -M/LC \\ -M/LC & \frac{1}{C} - \omega^2(L - M^2/L) \end{pmatrix} = 0 \qquad \text{etc.}$$

Double pendulum with spring (3):



- spring unstretched when pendula vertical
- small osc. limit
- spring unstretched length = D

$$\underline{r}_1 = \left(L\sin\varphi_1,\, L(1-\cos\varphi_1)\right) \qquad \underline{r}_2 = \left(D + L\sin\varphi_2,\, L(1-\cos\varphi_2)\right)$$

$$(\underline{r}_1-\underline{r}_2)^2 = D^2 + 2DL(\sin\varphi_2 - \sin\varphi_1) + L^2(\sin\varphi_1 - \sin\varphi_2)^2 + L^2(\cos\varphi_1 - \cos\varphi_2)^2$$

$$= D^2 + 2DL(\sin\varphi_2 - \sin\varphi_1) + 2L^2 - 2L^2\cos(\varphi_1 - \varphi_2)$$

$$\xrightarrow[|\varphi_i| \ll 1]{} D^2 + 2DL(\varphi_2-\varphi_1) + L^2(\varphi_1-\varphi_2)^2$$

$$\to \quad |\underline{r}_1 - \underline{r}_2| = \sqrt{\cdots} = D + L(\varphi_2-\varphi_1) + \cancel{\frac{L^2}{2D}(\varphi_1-\varphi_2)^2} - \cancel{\frac{L^2}{2D}(\varphi_1-\varphi_2)^2} + \ldots$$

$$\left(|\underline{r}_1-\underline{r}_2| - D\right)^2 = L^2(\varphi_2-\varphi_1)^2 + \text{higher order}$$

$\Rightarrow$ sort of obvious: at bottom of a circle x increases linearly
y " quadratically

$$\boxed{\sqrt{D^2+\varepsilon} = D\sqrt{1+\varepsilon/D^2} = D + \tfrac{1}{2}\varepsilon/D - \tfrac{1}{8}\varepsilon^2/D^2 + \ldots}$$

so $|\underline{r}_1 - \underline{r}_2| - D \simeq x_1 - x_2 - D$

$$\text{So } V = mg(y_1+y_2) + \tfrac{1}{2}k\left(|\underline{r}_1-\underline{r}_2| - D\right)^2$$

$$\simeq + \frac{mgL}{2}(\varphi_1^2+\varphi_2^2) + kL^2/2\,(\varphi_2-\varphi_1)^2$$

$$T = \frac{m}{2}\left(\dot{\underline{r}}_1^2 + \dot{\underline{r}}_2^2\right) = \frac{ml^2}{2}\left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2\right)$$

$$L = \frac{ml^2}{2}\left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2\right) - \frac{mgl}{2}\left(\varphi_1^2 + \varphi_2^2\right) - \frac{kl^2}{2}\left(\varphi_1 - \varphi_2\right)^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_i} = \frac{\partial L}{\partial \varphi_i} \rightarrow \quad ml^2\ddot{\varphi}_1 = -mgl\,\varphi_1 - kl^2(\varphi_1 - \varphi_2)$$

$$ml^2\ddot{\varphi}_2 = -mgl\,\varphi_2 + kl^2(\varphi_1 - \varphi_2)$$

look for $\varphi_1 = \phi_1 e^{-i\omega t}$ $\quad \varphi_2 = \phi_2 e^{-i\omega t}$

$$-m\cancel{l^2}\omega^2\phi_1 = -mg\cancel{l}\,\phi_1 - k\cancel{l^2}(\phi_1 - \phi_2)$$

$$-ml^2\omega^2\phi_2 = -mg\cancel{l}\,\phi_2 + k\cancel{l^2}(\phi_1 - \phi_2)$$

$$0 = \left(\omega^2 - \frac{g}{L} - \frac{k}{m}\right)\phi_1 + \frac{k}{m}\phi_2$$

$$0 = \left(\omega^2 - \frac{g}{L} - \frac{k}{m}\right)\phi_2 + \frac{k}{m}\phi_1$$

coupling

competing frequencies

Solution when det $= 0 = \left(\omega^2 - \frac{g}{L} - \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2$

$$= \left(\omega^2 - \frac{g}{L}\right)\left(\omega^2 - \frac{g}{L} - \frac{2k}{m}\right)$$

$\omega^2 = \frac{g}{L}$ – usual pendulum freq ; $0 = -\frac{k}{m}\phi_1 + \frac{k}{m}\phi_2 \rightarrow \phi_1 = \phi_2$

both pendula in phase, spring unstretched.

$\omega^2 = \frac{g}{L} + \frac{2k}{m}$ - new mode $\quad \theta = \frac{k}{m}\phi_1 + \frac{k}{m}\phi_2 \quad \phi_1 = -\phi_2$

pendula oscillate out of phase, stiffen due to spring

Try to transform to "normal mode" coordinates

let $\underline{\varphi} = \underline{\underline{A}}\,\underline{\theta}$, $\underline{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ $\underline{\underline{A}} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

$$L = \frac{1}{2}\underline{\dot{\varphi}}^T \underbrace{\begin{pmatrix} mL^2 & 0 \\ 0 & mL^2 \end{pmatrix}}_{\underline{\underline{M}}} \underline{\dot{\varphi}} - \frac{1}{2}\underline{\varphi}^T \underbrace{\begin{pmatrix} mgL + kL^2 & -kL^2 \\ -kL^2 & mgL + kL^2 \end{pmatrix}}_{\underline{\underline{V}}} \underline{\varphi}$$

$$\underline{\dot{\varphi}}^T \underline{\underline{M}}\,\underline{\dot{\varphi}} = \underline{\dot{\theta}}^T \underline{\underline{A}}^T \underline{\underline{M}}\,\underline{\underline{A}}\,\underline{\dot{\theta}} = \underline{\dot{\theta}}^T \begin{pmatrix} mL^2 & 0 \\ 0 & mL^2 \end{pmatrix} \underline{\dot{\theta}}$$

$$\underline{\varphi}^T \underline{\underline{V}}\,\underline{\varphi} = \underline{\theta}^T \underline{\underline{A}}^T \underline{\underline{V}}\,\underline{\underline{A}}\,\underline{\theta} = \ldots =$$

$$= \underline{\theta}^T \begin{pmatrix} mgL & 0 \\ 0 & mgL + 2kL^2 \end{pmatrix} \underline{\theta}$$

$$L \to \frac{1}{2}\left(mL^2\dot{\theta}_1^2 - mgL\,\theta_1^2\right) + \frac{1}{2}\left(mgL\,\dot{\theta}_2^2 - (mgL + 2kL^2)\,\theta_2^2\right)$$

: decoupled

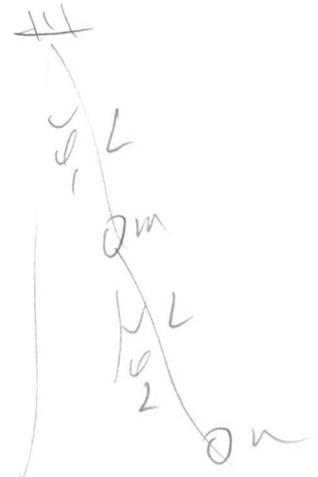
→ Can this be done in general?

Previously had $L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2)$ + quadratic

$\rightarrow$ diff mass $\frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 +$ "

but could also have $\dot{x}_1\dot{x}_2$ term:

Double pendulum



$$\underline{r_1} = (L\sin\varphi_1, L\cos\varphi_1) \simeq (L\varphi_1, L(1-\varphi_1^2/2))$$

$$\underline{r_2} = (L(\sin\varphi_1 + \sin\varphi_2), L(\cos \ldots))$$

$$\simeq (L(\varphi_1+\varphi_2), 2L - L(\varphi_1^2/2 + \varphi_2^2/2))$$

$$\dot{\underline{r_1}}^2 + \dot{r_2}^2 = L^2\dot{\varphi}_1^2 + L^2\cancel{(\varphi_1\dot{\varphi}_1)^2}$$

$$+ L^2(\dot{\varphi}_1+\dot{\varphi}_2)^2 + L^2\cancel{((\varphi_1+\varphi_2)(\dot{\varphi}_1+\dot{\varphi}_2))^2}$$

$$= 2L^2\dot{\varphi}_1^2 + 2L^2\dot{\varphi}_1\dot{\varphi}_2 + L^2\dot{\varphi}_2^2$$

So general $2+2$ case probably

$$L = \sum_{ij=1}^{2} \frac{1}{2} M_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}\sum_{ij} V_{ij}x_ix_j$$