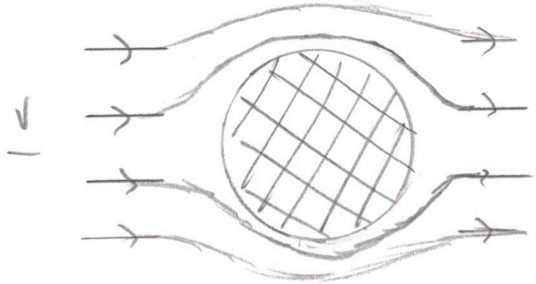
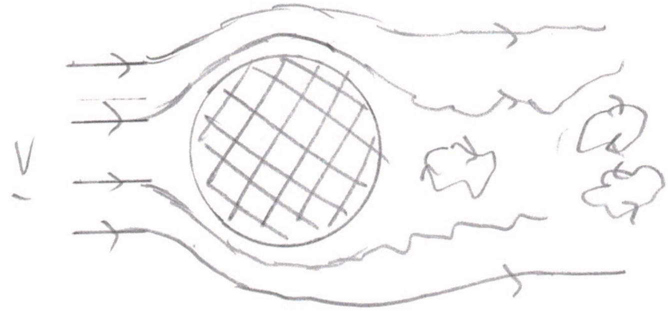
Air resistance (liquids also)

low speed –
laminar flow



$\underline{f} = -6\pi\mu R\underline{v} \equiv -b\underline{v}$

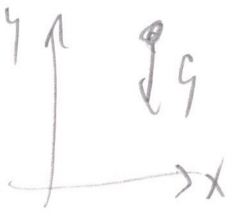
high speed –
unsteady/
turbulent flow



$\underline{f} = -c_D v^2 \hat{v}$

Really – smooth transition; low/high $\rightarrow Re \equiv \frac{\rho R v}{\mu} < 1 / > 1$

Laminar case 1st;



$m\dot{\underline{v}} = -b\underline{v} + m\underline{g}$

or $\begin{cases} m\dot{v}_x = -bv_x \\ m\dot{v}_y = -bv_y - mg \end{cases}$

x-eq easy $\dot{v}_x = -\frac{b}{m}v_x \rightarrow v_x(t) = v_x(0)\, e^{-bt/m}$

initial v_x decays due to air friction

$\frac{m}{b}$ = decay time scale, time for v_x to fall by 1/e.

v_y eq. has competing effects -

gravity balances drag at $v_y = -\frac{mg}{b} \equiv v_t$ "terminal velocity"

write $m\dot{v}_y = -bv_y - mg = -b\left(v_y - v_t\right)$

$\hookrightarrow m\frac{d}{dt}\left(v_y - v_t\right)$

so $\frac{d\left(v_y - v_t\right)}{v_y - v_t} = -\frac{b}{m}dt$ or $\log\left(v_y - v_t\right) = c_1 - \frac{bt}{m}$

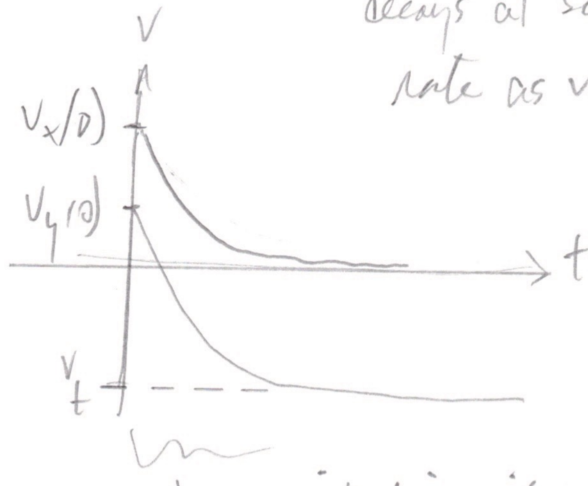
At $t=0$ $\log\left(v_y(0) - v_t\right) = c_1$ so

$$v_y(t) - v_t = \left(v_y(0) - v_t\right)e^{-bt/m}$$

or $$v_y(t) = v_y(0)\,e^{-bt/m} + v_t\left(1 - e^{-bt/m}\right)$$

decays at same rate as $v_x(0)$

asymptotes to v_t



transient time interval m/b

Then $y(t) = y(0) + \int_0^t dt'\, v_y(t')$ etc.

Direct method for $v(y)$:

use $\frac{dv_y}{dt} = \frac{dv_y}{dy} \cdot \frac{dy}{dt} = v_y \frac{dv_y}{dt}$

so $m v_y \frac{dv_y}{dy} = -b(v_y - v_t)$

$$\frac{v_y \, dv_y}{v_y - v_t} = \left(1 + \frac{v_t}{v_y - v_t}\right) dv_y = -\frac{b}{m} dy$$

$$\rightarrow \quad v_y(y) - v_y(0) + v_t \log \frac{v_y(y) - v_t}{v_y(0) - v_t} = -\frac{by}{m}$$

Nice feature - 2d problem reduces to 2 1-d problems

— x —

Non-linear case $\quad m\dot{\underline{v}} = -m\underline{g} - c_D \underbrace{\underline{v}^2 \hat{v}}$

mixes v_x & v_y

Does not split into 1-d problems in general -
look at special cases

Case 1 - motion in x-only

suppose $v_x(0) > 0$; $\quad m\dot{v}_x = -c_D v_x^2$

so $\frac{dv_x}{v_x^2} = -\frac{c_D}{m} dt \rightarrow \frac{1}{v_x(0)} - \frac{1}{v_x(t)} = -c_D t/m$

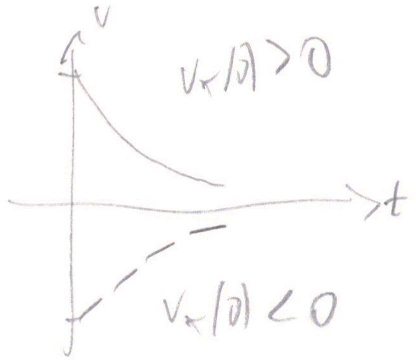
$$v_x(t) = \left[1/v_x(0) + c_D t/m\right]^{-1} \sim 1/t$$

What if $v_x(0) < 0$?

friction reverses, $c_D \to -c_D$

$$v_x(t) = \left[1/v_x(0) - c_D t/m \right]^{-1}$$

both terms < 0



True only for v_y "large", switches over to linear when v_y gets small

— x —

Case 2 motion in y only

$$m\dot{\underline{v}} = m\underline{g} - c_D v^2 \hat{\underline{v}} \quad \to \quad m\dot{v}_y = -mg \pm c_D v^2$$

use $-$ when $v_y > 0$

$+$ when $v_y < 0$

Apply to ball thrown upwards:

$v_y(0) > 0$ so initially $m\dot{v}_y = -mg - c_D v_y^2$

so $v_y \to 0$ at some y_{max}.

Then new problem – ball at rest at y_{max}, falls,

$m\dot{v}_y = -mg + c_D v_y^2$ ...

Details in Problem Set #1

Case 3 – no gravity $\quad m\dot{\underline{v}} = -C_D \underline{v}^2 \hat{v}$

take dot product with $\underline{v}$:

$$m\underline{v}\cdot\dot{\underline{v}} = \frac{m}{2}\frac{d}{dt}(\underline{v}\cdot\underline{v}) = \frac{m}{2}\frac{dv^2}{dt}$$

$$= -C_D \underline{v}^2 \underline{v}\cdot\hat{v} = -C_D v^3$$

let $u = v^2$; $\quad \frac{m}{2}\frac{du}{dt} = -C_D u^{3/2}$

so $\quad m u^{-1/2} = C_1 + C_D t$

or $\quad \frac{1}{v(t)} = \frac{1}{v(0)} + C_D t/m$

– same as case #1

General case $\quad m\dot{\underline{v}} = m\underline{g} \pm C_D (v_x^2 + v_y^2) \frac{\underline{v}}{\sqrt{v_x^2+v_y^2}}$

or $\quad m\dot{v}_x = \pm C_D \sqrt{v_x^2+v_y^2}\, v_x$

$$m\dot{v}_y = -mg \pm C_D \sqrt{v_x^2+v_y^2}\, v_y$$

no simple examples

Particle in a magnetic field $\quad m\dot{\underline{v}} = q\,\underline{v} \times \underline{B}$

take $\underline{B}$ = constant along $\hat{z}$ $= B\hat{z}$

$$\underline{v} \times \underline{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix}$$

$$= \hat{x}\left(v_y B\right) + \hat{y}\left(-v_x B\right) + \hat{z}\,(0)$$

so
$$\left.\begin{aligned} m\dot{v}_x &= qBv_y \\ m\dot{v}_y &= -qBv_x \end{aligned}\right\}\ \text{coupled ODEs}$$

$$m\dot{v}_z = 0 \longrightarrow v_z = \text{constant}$$

Method 1 - $\quad m\ddot{v}_x = qB\dot{v}_y = -\frac{q^2B^2}{m}v_x$

or $\quad \ddot{v}_x = -\omega^2 v_x \qquad \omega \equiv \frac{qB}{m} =$ "cyclotron freq."

↑ Harmonic oscillator equation

$$v_x = A\cos\omega t + B\sin\omega t$$

$$v_y = \frac{m}{qB}\dot{v}_x = \frac{1}{\omega}\left(-A\omega\sin\omega t + B\omega\cos\omega t\right)$$

$$= -A\sin\omega t + B\cos\omega t$$

So $\quad A = v_x(0), \quad B = v_y(0)$

Method 2 - let $v = v_x + i v_y$ $\quad i = \sqrt{-1}$

so $\quad \dot{v} = \dot{v}_x + i \dot{v}_y = \frac{qB}{m} v_y - i \frac{qB}{m} v_x$

$\quad = -i\omega v$

$\rightarrow \quad v(t) = v(0)\, e^{-i\omega t}$

$v_x(t) + i v_y(t) = \left(v_x(0) + i v_y(0)\right)\left(\cos\omega t - i \sin\omega t\right)$

Equate real + imaginary parts:

$$\left.\begin{aligned} v_x(t) &= v_x(0)\cos\omega t + v_y(0)\sin\omega t \\ v_y(t) &= v_y(0)\cos\omega t - v_x(0)\sin\omega t \end{aligned}\right\} \text{same thing}$$

Motion: $\quad x(t) = x(0) + \frac{v_x(0)}{\omega}\sin\omega t - \frac{v_y(0)}{\omega}\cos\omega t$

$\quad y(t) = y(0) + \frac{v_y(0)}{\omega}\sin\omega t + \frac{v_x(0)}{\omega}\cos\omega t$

so $\quad \left(x(t) - x(0)\right)^2 + \left(y(t) - y(0)\right)^2 = \frac{v_x(0)^2 + v_y(0)^2}{\omega^2}$

Also $\quad z(t) = z(0) + v_z(0)\, t$

- circle in x-y plane, linear along z

$\rightarrow$ helix

Add linear air resistance

$$m\dot{\underline{v}} = -b\underline{v} + q\,\underline{v}\times\underline{B}$$

so

$$m\dot{v}_x = -bv_x + qBv_y$$

$$m\dot{v}_y = -bv_y - qBv_x$$

$$m\dot{v}_z = -bv_z \longrightarrow v_z(t) = v_z(0)\,e^{-bt/m}$$

try method 1:

$$m\ddot{v}_x = -b\dot{v}_x + qB\dot{v}_y = -b\dot{v}_x + qB\left(-\frac{b}{m}v_y - \frac{qB}{m}v_x\right)$$

screws it up!

try method 2:

$$m(\dot{v}_x + i\dot{v}_y) = -b(v_x + iv_y) + qB(v_y - iv_x)$$

$$\underbrace{}_{\dot{\nu}} \qquad \underbrace{}_{\nu} \qquad \underbrace{}_{-i\nu}$$

or $\dot{\nu} = -(\beta + i\omega)\nu$ where $\beta = b/m$, $\omega = \frac{qB}{m}$

$$\nu(t) = \nu(0)\,e^{-(\beta + i\omega)t}$$

$$= e^{-\beta t} \times \text{previous result}$$

so $v_x(t) = e^{-\beta t}\left[v_x(0)\cos\omega t + \ldots\right]$

Method 3 - linear algebra

$$m\dot{v}_x = w v_y \qquad\qquad m\dot{v}_y = -w v_x$$

$$\rightarrow \underbrace{\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix}}_{\equiv \dot{\underline{v}}} = \begin{pmatrix} w v_y \\ -w v_x \end{pmatrix} = w \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\equiv \underline{\underline{J}}} \underbrace{\begin{pmatrix} v_x \\ v_y \end{pmatrix}}_{\underline{v}}$$

or $\dot{\underline{v}}(t) = w \underline{\underline{J}} \cdot \underline{v}(t)$

formal solution $\underline{v}(t) = \underbrace{e^{wt\underline{\underline{J}}}}_{\text{what is this?}} \cdot \underline{v}(0)$

Define $e^{\underline{\underline{M}}}$ for matrix $\underline{\underline{M}}$ as

$$= \underline{\underline{1}} + \underline{\underline{M}} + \frac{1}{2}\underline{\underline{M}}^2 + \frac{1}{3!}\underline{\underline{M}}^3 + \cdots$$

(can show this always converges)

so
$$\frac{d}{dt} e^{wt\underline{\underline{J}}} = \frac{d}{dt}\left(\underline{\underline{1}} + wt\underline{\underline{J}} + \frac{1}{2}(wt)^2\underline{\underline{J}}^2 + \frac{1}{3!}(wt)^3\underline{\underline{J}}^3 + \cdots\right)$$

$$= w\underline{\underline{J}} + w(wt)\underline{\underline{J}}^2 + \frac{1}{2}w(wt)^2\underline{\underline{J}}^3 + \cdots$$

$$= w\underline{\underline{J}}\left(\underline{\underline{1}} + wt\underline{\underline{J}} + \frac{1}{2}(wt)^2\underline{\underline{J}}^2 + \cdots\right)$$

$$= w\underline{\underline{J}}\, e^{wt\underline{\underline{J}}}$$

and
$$\frac{d}{dt}\left(e^{wt\underline{\underline{J}}} \cdot \underline{v}(0)\right) = w\underline{\underline{J}}\left(e^{wt\underline{\underline{J}}} \cdot \underline{v}(0)\right)$$

$\therefore$ solves ODE + initial condition

What is $e^{\omega t \underline{\underline{J}}}$? use the power series

$$\underline{\underline{J}}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\underline{\underline{1}}$$

$$\underline{\underline{J}}^3 = \underline{\underline{J}}^2 \cdot \underline{\underline{J}} = -\underline{\underline{J}}$$

$$\underline{\underline{J}}^4 = (\underline{\underline{J}}^2)^2 = +1 \quad \text{ok}$$

$$e^{\omega t \underline{\underline{J}}} = \underline{\underline{1}} + \omega t \underline{\underline{J}} - \frac{1}{2}(\omega t)^2 \underline{\underline{1}} - \frac{1}{3!}(\omega t)^3 \underline{\underline{J}} + \frac{1}{4!}(\omega t)^4 \underline{\underline{1}} \cdots$$

$$= \left(1 - \frac{1}{2}(\omega t)^2 + \frac{1}{4!}(\omega t)^4 - \cdots\right)\underline{\underline{1}}$$

$$+ \left(\omega t - \frac{1}{3!}(\omega t)^3 + \cdots\right)\underline{\underline{J}}$$

$$= \cos\omega t \cdot \underline{\underline{1}} + \sin\omega t \cdot \underline{\underline{J}}$$

$$e^{\omega t \underline{\underline{J}}} \cdot \underline{v}(0) = \cos\omega t \begin{pmatrix} v_x(0) \\ v_y(0) \end{pmatrix} + \sin\omega t \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_x(0) \\ v_y(0) \end{pmatrix}$$

$$\left.\begin{aligned} v_x(t) &= v_x(0)\cos\omega t + v_y(0)\sin\omega t \\ v_y(t) &= v_y(0)\cos\omega t - v_x(0)\sin\omega t \end{aligned}\right\} \text{same thing!}$$

What about friction?

$$\left.\begin{aligned} \dot{v}_x &= -\beta v_x + \omega v_y \\ \dot{v}_y &= -\beta v_y - \omega v_x \end{aligned}\right. \rightarrow \dot{\underline{v}} = \underbrace{\begin{pmatrix} -\beta & \omega \\ -\omega & -\beta \end{pmatrix}}_{\equiv \underline{\underline{K}} = -\beta \underline{\underline{1}} + \omega \underline{\underline{J}}} \underline{v}$$

solution is $\underline{v}(t) = e^{t\underline{\underline{K}}t}\underline{v}(0) = \underbrace{e^{-\beta t\underline{\underline{1}} + \omega t\underline{\underline{J}}}}_{\text{messy power series}} \cdot \underline{v}(0)$

but $e^{-\beta t\underline{\underline{1}} + \omega t\underline{\underline{J}}} = e^{-\beta t\underline{\underline{1}}}e^{\omega t\underline{\underline{J}}}$

because $\underline{\underline{1}}$ commutes with any matrix
" $e^{\alpha\underline{\underline{1}}}$ " "

so $\underline{v}(t) = e^{-\beta t\underline{\underline{1}}} \cdot \left(e^{\omega t\underline{\underline{J}}} \cdot \underline{v}(0)\right)$

$= \begin{pmatrix} e^{-\beta t} & 0 \\ 0 & e^{-\beta t} \end{pmatrix} \times$ previous $\underline{v}(t)$

so extra factor of $e^{-\beta t}$ in v_x, v_y.