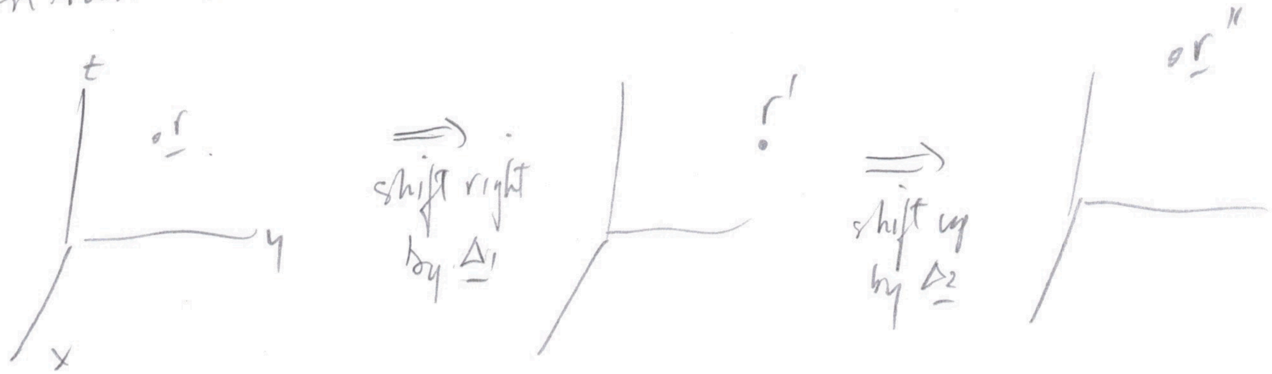
3d Rotations

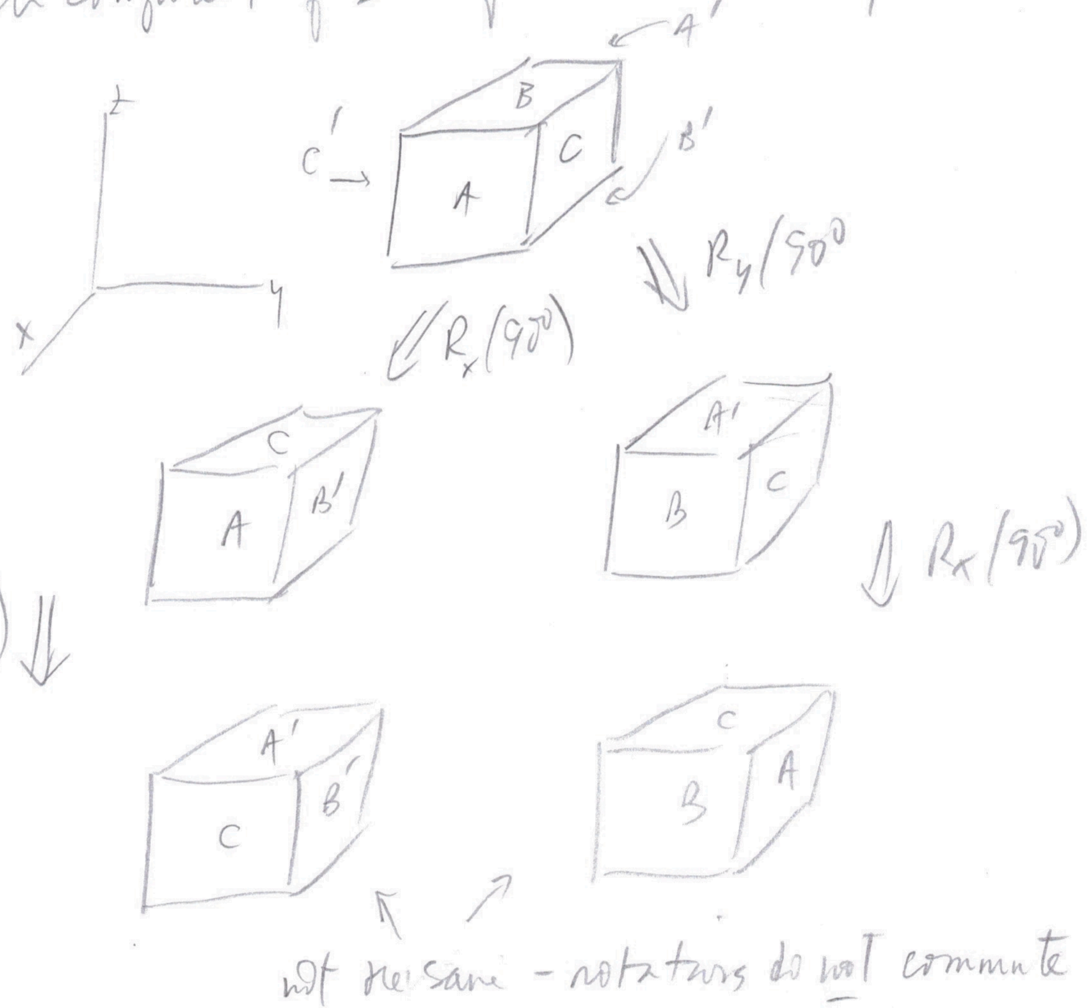
contrast to translations



$\underline{r}' = \underline{r} + \underline{\Delta}_1 \qquad \underline{r}'' = r' + \underline{\Delta}_2 = \underline{r} + \underline{\Delta}_1 + \underline{\Delta}_2$

→ { Same result if order reversed - translations "commute"
Each component of $\underline{r}$ shifts independently

rotation



not the same - rotations do not commute

Quantitative Argument:

write $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so $\underline{r} \xrightarrow{\text{rotate}} \underline{\underline{R}} \cdot \underline{r}$

where $\underline{\underline{R}}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ usual transformation

$$\left.\begin{aligned} \underline{\underline{R}}_y(\theta) &= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \\ \underline{\underline{R}}_x(\theta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \end{aligned}\right\}$$

permute indices
$x \to y \to z \to x \ldots$
$1 \to 2 \to 3 \to 1 \ldots$

Then $\underline{r} \to \underline{\underline{R}}_y\left(\frac{\pi}{2}\right) \cdot \left(\underline{\underline{R}}_x\left(\frac{\pi}{2}\right) \cdot \underline{r}\right) = \left(\underline{\underline{R}}_y \underline{\underline{R}}_x\right) \cdot \underline{r}$ 1st case

$\underline{r} \to \underline{\underline{R}}_x\left(\frac{\pi}{2}\right) \cdot \left(\underline{\underline{R}}_y\left(\frac{\pi}{2}\right) \cdot \underline{r}\right) = \left(\underline{\underline{R}}_x \underline{\underline{R}}_y\right) \cdot \underline{r}$

and $\underline{\underline{R}}_x \underline{\underline{R}}_y \neq \underline{\underline{R}}_y \underline{\underline{R}}_x$: finite rotations don't commute

But infinitesimal ones do:

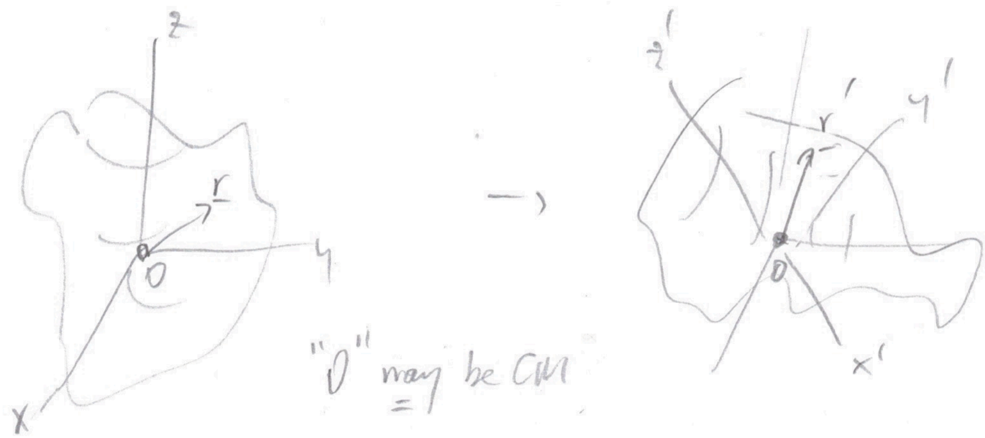
$$\underline{r} \xrightarrow[d\underline{\theta}_1]{} \underline{r} + d\underline{\theta}_1 \times \underline{r}$$

$$\xrightarrow[d\underline{\theta}_2]{} \left(\underline{r} + d\underline{\theta}_1 \times \underline{r}\right) + d\underline{\theta}_2 \times \left(\underline{r} + d\underline{\theta}_1 \times \underline{r}\right)$$

$$= \underline{r} + \underbrace{\left(d\underline{\theta}_1 + d\underline{\theta}_2\right)}_{\text{independent of order}} \times \underline{r} + \underbrace{O(d\theta^2)}_{\text{ignore}}$$

Finite rotations (for any axis):

Think of a coordinate system fixed in a solid body, which rotates about a fixed point "O".



"O" may be CM

old c-s $\underline{r} = x\hat{x} + y\hat{y} + z\hat{z}$ or $\underline{r} = \sum^{3} r_i \hat{e}_i$

after rotation $\underline{r} \to \underline{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$

also $= x\hat{x}' + y\hat{y}' + z\hat{z}'$

ie: $\underline{r}'$ = new components in old basis

= old components in rotated basis

Connection? $\hat{x}\cdot\underline{r}' = x' = (\hat{x}\cdot\hat{x}')\,x + (\hat{x}\cdot\hat{y}')\,y + (\hat{x}\cdot\hat{z}')\,z$

$\hat{y}\cdot\underline{r}' = y' = (\hat{y}\cdot\hat{x}')\,x + \cdots$

so new components are linear combinations of old ones.

Use 1-2-3 notation $r_1' = a_{11} r_1 + a_{12} r_2 + a_{13} r_3$ etc

$a_{ij} \equiv \hat{e}_i \cdot \hat{e}_j' = \cos\Delta\theta_{ij'}$ "direction cosine" or $r_i' = \sum_{j=1}^{3} a_{ij} r_j$

Matrix notation $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \text{etc.} & \end{pmatrix}$

$\rightarrow \underline{r}' = \underline{\underline{A}} \cdot \underline{r}$

Properties of $\underline{A}$: a_{ij} = real number, $|a_{ij}| \leq 1$

orthogonality – under rotation $\underline{r}^2 = \underline{r}'^2$

in matrix form, $\underline{r}^2 = r_1^2 + r_2^2 + r_3^2 = (r_1 \; r_2 \; r_3) \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$

$= \underline{r}^T \cdot \underline{r}$

so $\underline{r}'^2 = \underline{r}^T \underline{\underline{A}}^T \underline{\underline{A}} \underline{r} = \underline{r}^T \cdot \underline{r}$

$\uparrow r_i' = \sum_j A_{ij} r_j = \sum_j r_j (A^T)_{ji}$

Holds for all $\underline{r}$ so $\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{1}}$ = unit matrix $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

Also $\underline{\underline{A}}^{-1}$ exists, because $\det(A^T A) = (\det A)^2 = \det \underline{\underline{1}} = 1$

(physically – can go back + forth for $\underline{r}$, $\underline{r}'$)

so $\underline{\underline{A}}^T = (\underline{\underline{A}}^T \underline{\underline{A}}) \underline{\underline{A}}^{-1} = \underline{\underline{1}} \cdot \underline{\underline{A}}^{-1}$ or $\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$

$\rightarrow \underline{\underline{A}}$ = orthogonal matrix

Note: $\underline{r}$ = any vector fixed in a rigid body

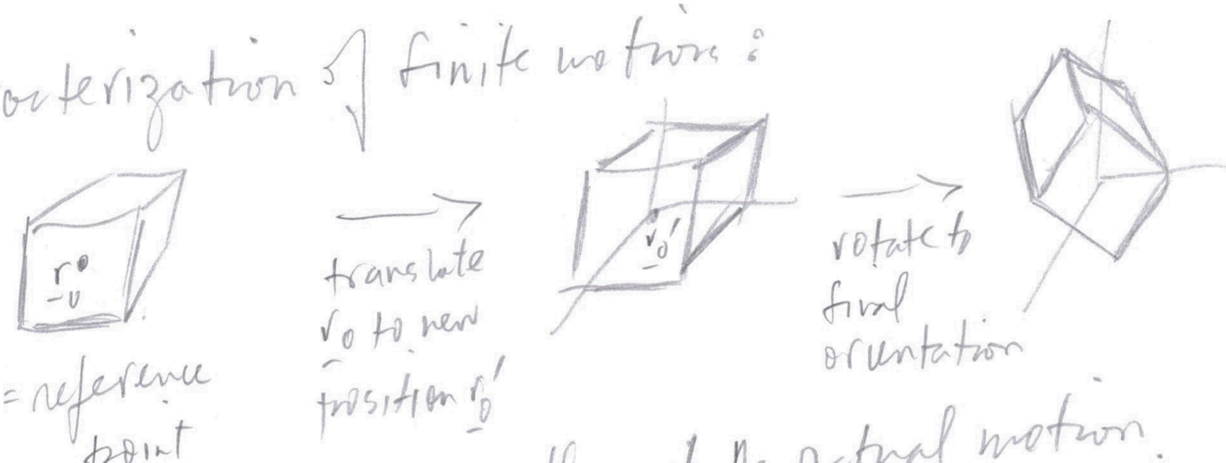
so $\dot{\underline{r}}, \ddot{\underline{r}}, \underline{L}, \underline{P}, \ldots$ have the same transformation law.

Rotations always preserve lengths in rigid bodies, therefore always described by an orthogonal matrix

Converse – is any orth. matrix equivalent to a rotation by some angle about some axis?

Yes – Euler's Theorem (proof in Goldstein et al. book)

Characterization of finite motion:



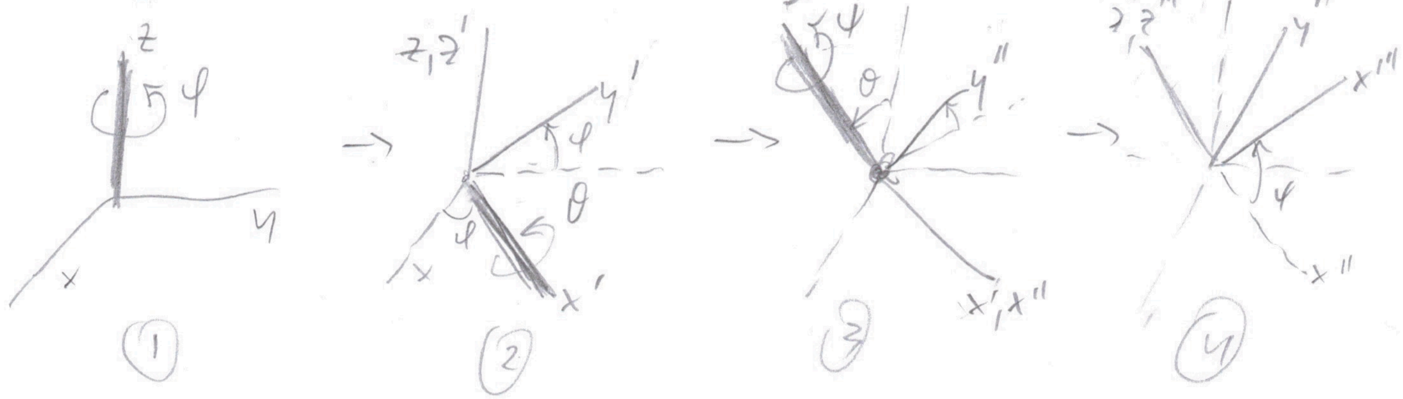
Usually not the actual motion.

How to specify orientation?

need 3 angles – 2 to give rotation axis (e.g. θ + φ)

1 for rotation about the axis

convention – "Euler angles"



Dynamics

Focus on motion about point "O" fixed in a rigid body

$\rightarrow$ masses m_i at $\underline{r}_0 + \underline{r}_i$; $\dot{\underline{r}}_i = \underline{w} \times \underline{r}_i$ since the motion wrt $\underline{r_0}$ is a rotation.

$$\underline{L} = \sum_i (\underline{r}_0 + \underline{r}_i) \times m_i (\dot{\underline{r}}_0 + \dot{\underline{r}}_i)$$

$$= \underline{r}_0 \times M \dot{\underline{r}}_0 + \underline{r}_0 \times \sum m_i \dot{\underline{r}}_i + \sum m_i \underline{r}_i \times \dot{\underline{r}}_0 + \sum m_i \underline{r}_i \times \dot{\underline{r}}_i$$

$$T = \sum_i \tfrac{1}{2} m_i (\dot{\underline{r}}_0 + \dot{\underline{r}}_i)^2$$

$$= \tfrac{1}{2} M \dot{\underline{r}}_0^2 + \dot{\underline{r}}_0 \cdot \sum m_i \dot{\underline{r}}_i + \tfrac{1}{2} \sum m_i \dot{\underline{r}}_i^2$$

In the coordinate system where $\underline{r_0}$ is fixed at the origin

$$\underline{L} \rightarrow \sum_i m_i \underline{r}_i \times (\underline{w} \times \underline{r}_i) = \sum_i m_i \left[\underline{w}\, r_i^2 - \underline{r}_i (\underline{w} \cdot \underline{r}_i) \right]$$

$$T \rightarrow \tfrac{1}{2} \sum m_i (\underline{w} \times \underline{r}_i)^2 = \sum_i \tfrac{1}{2} m_i \left[\underline{w}^2 \underline{r}_i^2 - (\underline{w} \cdot \underline{r}_i)^2 \right]$$

(If O accelerates, get extra fictitious forces which affect the CM motion.)

Special case: $\underline{w} = \text{constant} = w\hat{z}$:

$$L_z = w \sum_i m_i (r_i^2 - z_i^2) = w \sum_i m_i (x_i^2 + y_i^2) = w I_{zz}$$

$$T = \tfrac{1}{2} \left(\sum_i m_i (x_i^2 + y_i^2) \right) w^2 = \tfrac{1}{2} I_{zz} w^2$$ studied earlier

General case:

$$L_\alpha = \sum_i m_i \left[\omega_\alpha \underline{r}_i^2 - r_{\alpha i}\left(\sum_\beta r_{\beta i}\omega_\beta\right)\right]$$ $r_{\alpha i}$ - α^{th} component of $\underline{r}_i$, $\alpha = 1,2,3$

$$= \sum_\beta \left(\sum_i m_i \left(\delta_{\alpha\beta} \underline{r}_i^2 - r_{\alpha i} r_{\beta i}\right)\right) \omega_\beta$$

$$\equiv \sum_\beta I_{\alpha\beta}\,\omega_\beta \quad \text{or} \quad \underline{L} = \underline{\underline{I}} \cdot \underline{\omega} \quad ; \quad \underline{L} \text{ not } \parallel \text{ to } \underline{\omega}$$

$$T = \sum_i \tfrac{1}{2} m_i \left(\underline{\omega}^2 r_i^2 - \sum_\alpha \omega_\alpha r_{\alpha i} \cdot \sum_\beta \omega_\beta r_{\beta i}\right)$$

$$= \frac{1}{2} \sum_{\alpha,\beta} \left(\sum_i m_i \delta_{\alpha\beta} - r_{\alpha i} r_{\beta i}\right) \omega_\alpha \omega_\beta$$

$$= \frac{1}{2} \sum_{\alpha\beta} I_{\alpha\beta}\,\omega_\alpha \omega_\beta = \frac{1}{2}\, \underline{\omega} \cdot \underline{\underline{I}} \cdot \underline{\omega}$$

$\underline{\underline{I}} = \left(I_{\alpha\beta}\right)$ = "moment of inertia $\underline{\underline{\text{tensor}}}$"

Then Lagrangian $L = T - V$, V = fn of $\{\underline{r}_i\}$

& the torque eq is $\dfrac{d\underline{L}}{dt} = \underline{\Gamma}$

so $\underline{\underline{I}}$ controls the motion of the rigid body.

In detail, $\dfrac{d}{dt}\left(\underline{\underline{I}}(t) \cdot \underline{\omega}(t)\right) = \underline{\Gamma}(t)$

or $\underline{\underline{I}} \cdot \dot{\underline{\omega}} + \dot{\underline{\underline{I}}} \cdot \underline{\omega} = \underline{\Gamma}$

awkward t-dependence - see later

Computing $\underline{\underline{I}}$:

trivial for discrete mass points $\{(x_i, y_i, z_i);\ i = 1, 2 \cdots N\}$

$$I_{\alpha\beta} = \sum_i m_i \left(\underline{r}_i^2 \delta_{\alpha\beta} - r_{i\alpha} r_{i\beta} \right)$$

$$\Rightarrow \quad I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$$

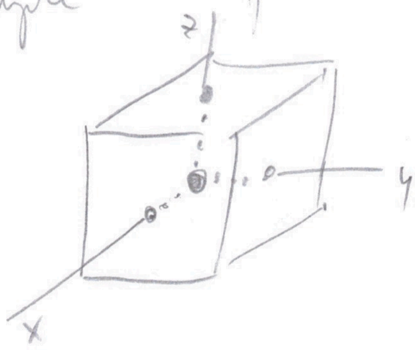
$$I_{xy} = \sum_i m_i (-x_i y_i) \quad \text{etc.}$$

for continuous distributions (solids) use

$$I_{\alpha\beta} = \int d^3r\, \rho(\underline{r}) \left(\underline{r}^2 \delta_{\alpha\beta} - r_\alpha r_\beta \right)$$

density

Example - uniform cube of side a, about CM



$$M = \rho a^3$$

$$I_{xx} = \int_{\text{cube}} \rho\, d^3r\, (y^2 + z^2)$$

$$I_{xy} = \int_{\text{cube}} \rho\, d^3r\, (-xy)$$

Other $I_{\alpha\beta}$ are the same; $I_{yy} = I_{zz} = I_{xx}$

$$I_{yz} = I_{zx} = I_{xy}$$

$$I_{xx} = \rho \int_{-a/2}^{+a/2} dx\,dy\,dz\, (y^2) + \rho \int dx\,dy\,dz\, z^2$$

$$= 2 \cdot \rho \cdot \frac{a}{2} \cdot 2\frac{1}{3}\left(\frac{a}{2}\right)^3 = \rho a^5/6 = Ma^2/6$$

$$I_{xy} = -\rho \int_{-a/2}^{a/2} dx\,dy\,dz\; xy = -\rho \cdot 0 \cdot 0 \cdot a = 0$$

so $\underline{\underline{I}} = \begin{pmatrix} Ma^2/6 & & 0 \\ & Ma^2/6 & \\ 0 & & Ma^2/6 \end{pmatrix}$ and $\underline{L} = \frac{Ma^2}{6}\underline{\omega}$

What about $\underline{\underline{I}}$ about a corner?

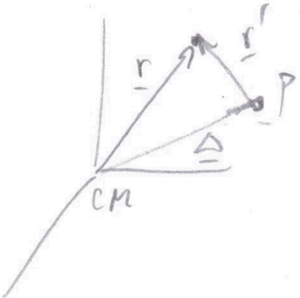
→ Parallel-axis theorem:

start from $I_{CM,\alpha\beta} = \int \rho\, d^3r \left(\underline{r}^2 \delta_{\alpha\beta} - r_\alpha r_\beta\right)$

$\underline{r}$ = vector from CM to point $\underline{r}$ in body

for any other point P

$$I'_{\alpha\beta} = \int \rho\, d^3r' \left(\underline{r}'^2 \delta_{\alpha\beta} - r'_\alpha r'_\beta\right)$$



$\underline{r}'$ = vector from P to point $\underline{r}'$ in body

$\underline{r}' = \underline{r} - \underline{\Delta}$ $\quad \underline{\Delta}$ = vector from CM to P

so $I'_{\alpha\beta} = \int \rho\, d^3r \left[(\underline{r}-\underline{\Delta})^2 \delta_{\alpha\beta} - (r-\Delta)_\alpha (r-\Delta)_\beta \right]$

$$= \int \rho\, d^3r \Big[\underline{r}^2 \delta_{\alpha\beta} - r_\alpha r_\beta$$

$$\underbrace{-2\,\underline{r}\cdot\underline{\Delta}\,\delta_{\alpha\beta} + r_\alpha \Delta_\beta + r_\beta \Delta_\alpha}_{\to 0}$$

$\int d^3r\, r_\alpha = 0$: def of CM

$$+ \underline{\Delta}^2 \delta_{\alpha\beta} - \Delta_\alpha \Delta_\beta \Big]$$

$$= I_{CM,\alpha\beta} + M\left(\underline{\Delta}^2 \delta_{\alpha\beta} - \Delta_\alpha \Delta_\beta\right)$$

for the cube: $\underline{\Delta}$ = vector from CM to corner

$$= \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right) \quad \text{say}$$

so $$I'_{xx} = I_{xx} + M\left(3 \cdot \left(\frac{a}{2}\right)^2 - \frac{a}{2} \cdot \frac{a}{2}\right)$$

$$= \frac{1}{6} Ma^2 + \frac{Ma^2}{2} = \frac{2}{3} Ma^2$$

$$I'_{xy} = I_{xy} + M\left(0 - \frac{a}{2} \cdot \frac{a}{2}\right) = -Ma^2/4$$

etc. - same symmetry because all 3 Δ_α the same

$$\rightarrow \quad \underline{\underline{I}} = \begin{pmatrix} I_0 & I_1 & I_1 \\ \cdot & I_0 & I_1 \\ \text{sym} & & I_0 \end{pmatrix} \quad I_0 = \frac{2}{3} Ma^2, \; I_1 = -\frac{1}{4} Ma^2$$

Effect on $\underline{L}$:

$\underline{w} = (w, 0, 0)$ about CM: $$\underline{L}_{cm} = \underline{\underline{I}}_{cm} \cdot \underline{w}$$

$$= \left(\frac{Ma^2}{6} w, 0, 0\right)$$

$$= \frac{Ma^2}{6} \underline{w}$$

about corner $$\underline{L}' = \underline{\underline{I}}' \cdot \underline{w} = (I_0 w, I_1 w, I_1 w)$$

not $\parallel$ to $\underline{w}$

$w = (1,1,1)$ $$\underline{L}_{cm} = \frac{Ma^2}{6} \underline{w} \quad \text{again}$$

$$\underline{L}' = (I_0 + 2I_1) \underline{w} = \frac{Ma^2}{6} \underline{w} \quad \text{again!}$$

$\rightarrow$ Highly symmetric, same as a sphere.