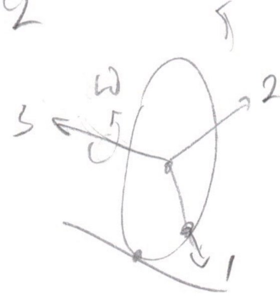
<u>Example</u> A wheel of radius R rolls in a straight line on the ground w/o slipping at velocity v_0. What's the acceleration of a point on the rim wrt ground?

Take a system rotating with the wheel (non-inertial) where a point is $\underline{r} = R\hat{e}_1$ say; $\dot{\underline{r}} = \ddot{\underline{r}} = 0$.

In the ground frame this point has velocity

$$\underbrace{\left.\frac{d\underline{r}}{dt}\right|_{grnd}}_{\text{inertial}} = \underbrace{\left.\frac{d\underline{r}}{dt}\right|_{rot}}_{\text{non-inertial}} + \underline{\omega} \times \underline{r} = \underline{\omega} \times (R\hat{e}_1) = R\hat{e}_2$$



and acceleration

$$\underline{a}_g = \left.\frac{d^2\underline{r}}{dt^2}\right|_{grnd} = \left(\left.\frac{d}{dt}\right|_{rot} + \underline{\omega}\times\right)\left.\frac{d\underline{r}}{dt}\right|_{grnd}$$

$$= \underline{\omega} \times (\underline{\omega} \times R\hat{e}_1)$$

Now $\underline{\omega} = \omega\hat{e}_3 = \omega\hat{z}$, $\omega = v_0/R$

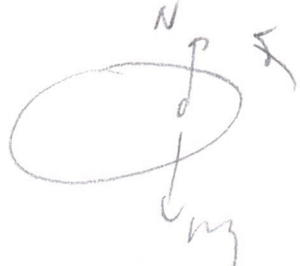
so $\underline{a}_g = \omega^2 R\hat{e}_3 \times (\underbrace{\hat{e}_3 \times \hat{e}_1}_{\hat{e}_2})$

$$= \underbrace{-\omega^2 R\hat{e}_1}$$

: vector from fixed point on rim towards centre of wheel

Example: A mass m "sits" on a frictionless turntable which rotates at ω. What forces act on it?

ground frame: $\underline{F} = m\underline{g} - \underline{N} = 0$: it's at rest

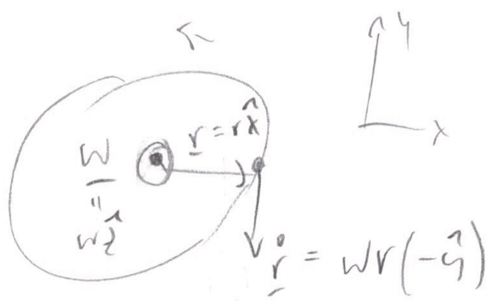


no force in plane, no motion.

turntable frame: still have $\underline{N} + m\underline{g}$, but also

$$F_{cor} = -2m\,\underline{\omega} \times \dot{\underline{r}}$$

$$F_{cent} = -m\,\underline{\omega} \times (\underline{\omega} \times \underline{r})$$



$$F_{cor} = -2m\omega\dot{r}\,\hat{x} = -2m\omega^2 r\,\hat{x}$$

$$F_{cent} = +m\omega^2 r\,\hat{x}$$

$$\underline{F}_{total} = -m\omega^2 r\,\hat{x} \quad \checkmark$$

Opposite limit: mass fixed in place on the turntable

ground frame: rotate in a circle, centripetal force $-m\omega^2\underline{r}$

turntable frame: centrifugal force $+m\omega^2\underline{r}$ balanced by friction

no Coriolis force

Quantitative analysis of this (9.20 in book)

frictionless puck on horizontal rotating table

(a) Turntable frame $m\ddot{\underline{r}} = -2m\,\underline{\omega}\times\dot{\underline{r}} - m\underline{\omega}\times(\underline{\omega}\times\underline{r})$

take $\underline{\omega} = \omega\hat{z}$ + $\underline{r} = (x, y, 0)$

$$\underline{\omega}\times\dot{\underline{r}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = \hat{y}(\omega\dot{x}) + \hat{x}(-\omega\dot{y})$$

$$\underline{\omega}\times(\underline{\omega}\times\underline{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ -\omega y & \omega x & 0 \end{vmatrix} = \hat{x}(-\omega^2 x) + \hat{y}(-\omega^2 y)$$

So $\ddot{x} = 2\omega\dot{y} + \omega^2 x$

$\ddot{y} = -2\omega\dot{x} + \omega^2 y$

(b) let $\eta = x + iy \rightarrow \ddot{\eta} = -2i\omega\dot{\eta} + \omega^2\eta$

look for $\eta(t) \propto e^{-i\alpha t} \rightarrow -\alpha^2 = \omega^2 - 2\alpha\omega$

$\rightarrow$ single root $\alpha = \omega$, 2nd solution is $t e^{-i\omega t}$

So $\eta(t) = (A + Bt)e^{-i\omega t}$, A, B = complex constants

(c) If at $t=0$ $\underline{r} = (x_0, 0, 0)$ + $\dot{\underline{r}} = (v_{0x}, v_{0y}, 0)$

then $A = x_0$, $v_{0x} + iv_{0y} = B - i\omega A$

and $\eta(t) = [x_0 + v_{0x}t + v_{0x}t + i(v_{0y} + \omega x_0)t]e^{-i\omega t}$

or $x(t) = (x_0 + v_{0x}t)\cos\omega t + (v_{0y} + \omega x_0)t\sin\omega t$

$y(t) = -(x_0 + v_{0x}t)\sin\omega t + (v_{0y} + \omega x_0)t\cos\omega t$

(d) Behavior at large t:

$$x(t) \to t\left(v_{0x}\cos\omega t + (v_{0y}+\omega x_0)\sin\omega t\right)$$

$$y(t) \to t\left(-v_{0x}\sin\omega t + (v_{0y}+\omega x_0)\cos\omega t\right)$$

let $\sin\alpha = \dfrac{v_{0x}}{\sqrt{v_{0x}^2+(v_{0y}+\omega x_0)^2}}$, $\cos\alpha = \dfrac{v_{0y}+\omega x_0}{\sqrt{\cdots}}$

then $x(t) = \sqrt{\cdots}\,\sin(\omega t+\alpha)$

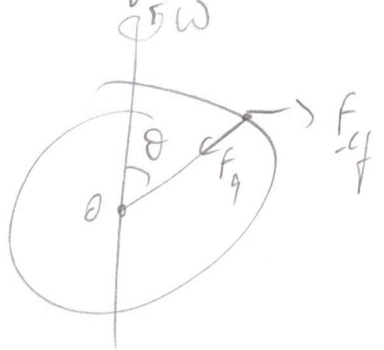
$y(t) = -\sqrt{\cdots}\,\cos(\omega t+\alpha)$

So this is motion on a circle of radius

$R = t\sqrt{v_{0x}^2+(v_{0y}+\omega x_0)^2}$ = outward spiral

Effects of rotational "extra" forces on motion on the Earth

1. Free fall with centrifugal force (simplest)



θ = co-lattitude

O = point to which g would aim w/o rotation; center of Earth if its spherical + uniform

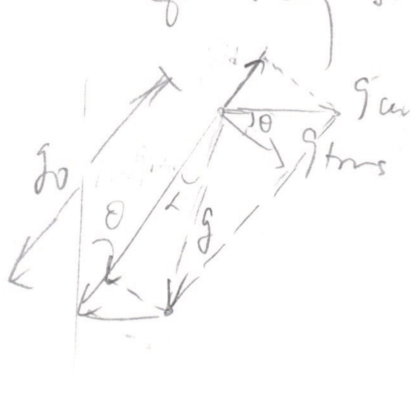
$$F_g = -m g_0 \hat{r}$$

$$F_{cf} = -m\,\omega \times (\omega \times r) = m\omega^2 R \sin\theta\, \hat{\rho} \quad \text{as above}$$

NB $\omega^2 R \xrightarrow[\text{Earth}]{} \left(\dfrac{2\pi}{24 \times 3600\,s}\right)^2 (6.4 \times 10^6\,m)$

$$\simeq 0.032\ m/s^2 \ll g_0 = 9.8\ m/s^2$$

So: motion deflected from straight down by an amt depending on lattitude. At what angle?



$$g_{rad} = g_0 - \omega^2 R \sin\theta \cdot \sin\theta$$

$$\simeq g_0$$

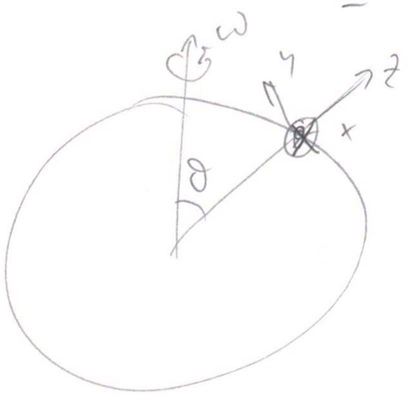
$$g_{trans} = \omega^2 R \sin\theta \cdot \cos\theta$$

$$\tan\alpha = \frac{g_{trans}}{g_{radial}} \simeq \frac{\omega^2 R \sin 2\theta}{2g} \lesssim 0.1 \text{ degrees}$$

2. Effects of Coriolis force

$$m\ddot{\underline{r}} = \underbrace{m\underline{g}_0 + \underline{F}_{cf}}_{m\underline{g}} + \underbrace{\underline{F}_{cor}}_{-2m\underline{\omega}\times\dot{\underline{r}}}$$

so $\ddot{\underline{r}} = \underline{g} - 2\underline{\omega}\times\dot{\underline{r}}$



choose local coord system at surface at θ

$\hat{z}$ = up = along $\underline{g}$
$\hat{y}$ = north
$\hat{x}$ = east

$\underline{\omega} = (0, \omega\sin\theta, \omega\cos\theta)$

up/down wrt $\underline{g}$ not $\underline{g}_0$

$\dot{\underline{r}} = (\dot{x}, \dot{y}, \dot{z})$

$\underline{g} = (0, 0, g)$

$$\underline{\omega}\times\dot{\underline{r}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \omega\sin\theta & \omega\cos\theta \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} = \hat{x}\,\omega(\dot{z}\sin\theta - \dot{y}\cos\theta) + \hat{y}(\omega\dot{x}\cos\theta) + \hat{z}(-\omega\dot{x}\sin\theta)$$

so
$$\ddot{x} = 2\omega(\dot{y}\cos\theta - \dot{z}\sin\theta)$$
$$\ddot{y} = -2\omega\dot{x}\cos\theta$$
$$\ddot{z} = -g + 2\omega\dot{x}\sin\theta$$

These are (solvable) linear eqs but it's tedious; instead use "regular perturbation theory"

Suppose $\dot{x}(t) = f(x) + w\,g(t)$ & w is small.

write $x(t) = x_0(t) + w x_1(t) + w^2 x_2(t) + \cdots$ & substitute

$$\dot{x}_0 + w\dot{x}_1 + \underset{+\cdots}{w^2 \dot{x}_2} = f(x_0 + w x_1 + w^2 x_2 \cdots)$$
$$+ w\, g(x_0 + w x_1 + w^2 x_2 \cdots)$$
$$= f(x_0) + f'(x_0)(w x_1 + w^2 x_2 \cdots) + \tfrac{1}{2} f''(x_0)(w x_1 \cdots)^2$$
$$+ w\left[g(x_0) + g'(x_0)(w x_1 + \cdots)^2 + \cdots\right]$$
$$= f(x_0) + w\left[f'(x_0)x_1 + g(x_0)\right] + w^2\left[f'(x_0)x_2 + \right.$$
$$\left. + \tfrac{1}{2} f''(x_0)x_1^2 + g'(x_0)x_1\right] + \cdots$$

then collect powers of w:

$\dot{x}_0 = f(x_0)$ — easy to solve, often

$\dot{x}_1 = x_1 f'(x_0) + g(x_0)$ — this linear, known coeff.

$\vdots$

etc.

Here write x, y, t as series in w^n & subst.

$\downarrow$

$$\begin{cases} \ddot{x}_0 + \omega \ddot{x}_1 + \cdots = 2\omega\cos\theta\,(\dot{y}_0 + \omega \dot{y}_1 + \cdots) \\ \qquad\qquad -2\omega\sin\theta\,(\dot{z}_0 + \omega\dot{z}_1 + \cdots) \\ \ddot{y}_0 + \omega \ddot{y}_1 + \cdots = -2\omega\cos\theta\,(\dot{x}_0 + \omega\dot{x}_1 + \cdots) \\ \ddot{z}_0 + \omega\ddot{z}_1 + \cdots = -g + 2\omega\sin\theta\,(\dot{x}_0 + \omega\dot{x}_1 + \cdots) \end{cases}$$

So $\ddot{x}_0 = \ddot{y}_0 = 0 \quad \ddot{z}_0 = -g \qquad O(\omega^0)$

$$\left.\begin{cases} \ddot{x}_1 = 2(\cos\theta\,\dot{y}_0 - \sin\theta\,\dot{z}_0) \\ \ddot{y}_1 = -2\cos\theta\,\dot{x}_0 \\ \ddot{z}_1 = 2\sin\theta\,\dot{x}_0 \qquad \text{etc.} \end{cases}\right\} O(\omega^1)$$

1. Free fall: $x(0) = y(0) = 0$; $z(0) = h$; $\underline{v}(0) = 0$

$O(\omega^0)$: $x_0 = y_0 = 0$; $z_0(t) = h - \frac{1}{2}gt^2$

$O(\omega^1)$
$$\ddot{x}_1 = +2gt\sin\theta$$
$$\ddot{y}_1 = \ddot{z}_1 = 0 \qquad x_1(t) = \frac{1}{3}gt^3\sin\theta$$

So to this order $\begin{cases} x(t) = \frac{1}{3}g\omega t^3\sin\theta \\ y(t) = z(t) = 0 \end{cases}$

$\rightarrow$ falling mass drifts to the East.

2. Thrown ball (problem 9.26, needed for HW)

same ODE set but $\underline{r}(0) = 0$, $\dot{\underline{r}}(0) = \underline{v_0}$

$O(\omega^0)$: $\quad \ddot{x}_0 = \ddot{y}_0 = 0 \quad \ddot{z}_0 = -g$

$\rightarrow \quad x_0(t) = v_{0x} t \quad y_0(t) = v_{0y} t \quad z_0(t) = -\frac{1}{2} g t^2 + v_{0z} t$

$O(\omega^1)$:

$$\ddot{x}_1 = 2\cos\theta \cdot v_{0y} - 2\sin\theta\,(-gt + v_{0z})$$

$$\ddot{y}_1 = -2\cos\theta\,(v_{0x})$$

$$\ddot{z}_1 = 2\sin\theta\,(v_{0x}) \quad \leftarrow \text{no } -g \text{ term - it's in } z_0$$

NB: BC here are $\underline{r}_1(0) = \dot{\underline{r}}_1(0) = 0$ because the $O(\omega^1)$ terms have already "eaten" the starting values.

So

$$x_1(t) = \cancel{2}(\cos\theta \cdot v_{0y} - \sin\theta \cdot v_{0z})\,t^2 + \frac{1}{3}\sin\theta\, g t^3$$

$$y_1(t) = -\cos\theta\, v_{0x}\, t^2$$

$$z_1(t) = \sin\theta\, v_{0x}\, t^2$$

Then

$$\underline{r}(t) = \underline{r}_0(t) + \omega\, \underline{r}_1(t) + \dots$$

$$\simeq \Big(v_{0x} t + 2\omega(\cos\theta\, v_{0y} - \sin\theta\, v_{0z})\,t^2 + \frac{1}{3}\omega g \sin\theta\, t^3,$$

$$v_{0y} t - \omega\cos\theta\, v_{0x}\, t^2,\ -\frac{1}{2} g t^2 + v_{0z} t + \omega \sin\theta\, v_{0x}\, t^2 \Big)$$

Numerical illustration (9.28):

shell fired due east at angle α above horizontal at co-lattitude

(a) w/o Earth's rotation $x(t) = v_0 \cos\alpha\, t$, $y(t) = 0$, $z = v_0 \sin\alpha\, t - \frac{1}{2} g t^2$

and when it lands $(z = 0)$ $t^* = \dfrac{2v_0 \sin\alpha}{g}$, $x^* = v_0 \cos\alpha\, t^*$

If $v_0 = 500$ m/s & $\alpha = 20°$, $t^* = 34.9$s, $x^* = 16.4$ km

(b) with the Earth's rotation and "lattitude 50° North" $(\theta = 40°)$

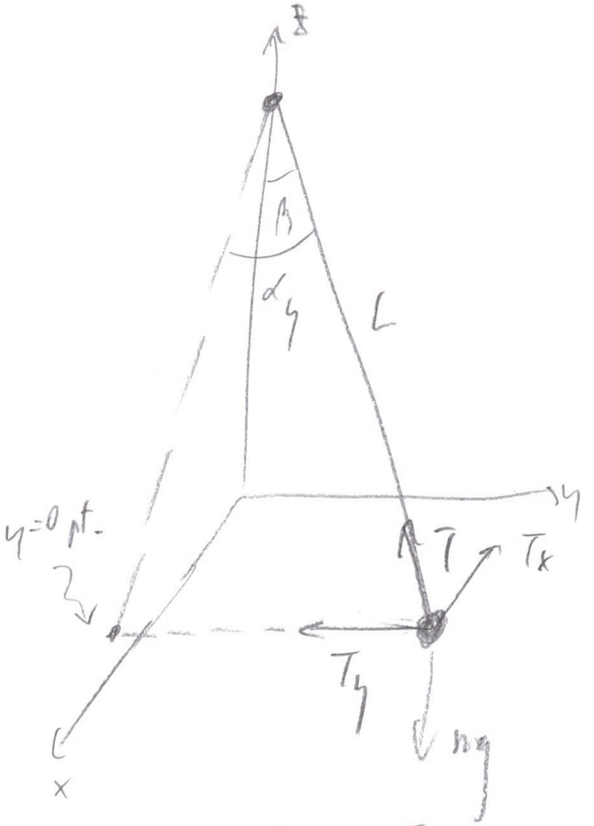
$y^* = \cancel{v_{0y}} t - \Omega v_{0x} \cos\theta\, t^2$

Since the corrections to z are $O(\Omega)$ the shell lands when $t = t^* + O(\omega)$, so to leading order & $\theta = 40°$

$y^* = -\Omega v_{0x} \cos\theta\, t^{*2} = -32$ m (South)

(c) If instead the shell is fired at "lattitude 50° South" then $\theta = 140°$ & since $\cos 40° = -\cos 140°$ the deflection is $+32$ m (North)

Foucault pendulum – moves in 3d with gravity + non-inertial forces



β = angle btw mass and "up"

$\alpha_{x,y}$ = Δ-angles as indicated

$$m\ddot{\underline{r}} = \underline{T} + m\underline{g}_0 - m\underline{\omega}\times(\underline{\omega}\times\underline{r}) - 2m\underline{\omega}\times\dot{\underline{r}}$$

$$= \underline{T} + m\underline{g} - 2m\underline{\omega}\times\dot{\underline{r}}$$

Here $T_y = -T\sin\alpha_y = -T\,y/L$

$T_x = -T\,x/L$

$T_z = \sqrt{T^2 - T_x^2 - T_y^2} = T + O\left(\frac{x^2}{L^2}, \frac{y^2}{L^2}\right)$

$z = \sqrt{L^2 - x^2 - y^2} = L + O\left(\frac{x^2}{L}, \frac{y^2}{L}\right) \simeq L$

So $m\ddot{z} \simeq 0 = T - mg + 0$ or $T \simeq mg$

Coriolis term is the same as before

$\rightarrow$ $\ddot{x} = -g\frac{x}{L} + 2\omega(\dot{y}\cos\theta - \dot{z}\sin\theta)$

$\ddot{y} = -g\frac{y}{L} - 2\omega\dot{x}\cos\theta$

$\dot{z} + \ddot{z} \simeq 0$

For $\omega \to 0$ get usual pendulum eqn in x and y

To solve, let $\xi = x + iy$

$$\ddot{\xi} = -\frac{g}{L}\xi + 2\omega(\dot{y} - i\dot{x})\cos\theta \quad \text{neglecting } \dot{z}$$

or $$\ddot{\xi} + 2i\underbrace{\omega\cos\theta}_{=\omega_z}\dot{\xi} + \omega_0^2 = 0 \quad \omega_0^2 = \frac{g}{L}$$

Look for $\xi \sim e^{i\omega t} \rightarrow -\omega^2 - 2\omega\omega_z + \omega_0^2 = 0$

or $$\omega = -\omega_z \pm \sqrt{\omega_z^2 + \omega_0^2}$$

$$\cong -\omega_z \pm \omega_0$$: since ω_z is small + want to observe ω_0 effects.

So $$\xi = e^{-i\omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right)$$

Take $x = A$, $y = 0$, $\dot{\underline{r}} = 0$ at $t = 0$

$$\rightarrow \quad \xi = A\cos\omega_0\, e^{-i\omega_z t}$$

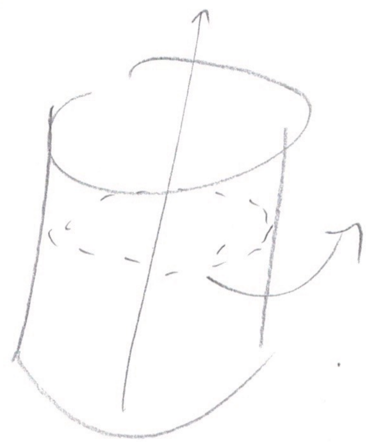
then $$r = \sqrt{x^2 + y^2} = |\xi| = A\cos\omega_0 t$$

$$\varphi = \tan^{-1}\frac{y}{x} = \tan^{-1}\underbrace{\frac{\text{Im}\,\xi}{\text{Re}\,\xi}}_{\tan\omega_z t} = -\omega_z t$$

So bob moves back + forth in a plane rotating at ω_z

Since $\omega_z = \omega\cos\theta$ can use this to determine $\theta \rightarrow$ lattitude

Application: rotating cup of liquid



liquid at radius ρ from centre feels centrifugal acceleration

$$\underline{a} = \omega^2 \rho \quad \text{(outward)}$$

+ gravitational force $F = mg$ (down)

which means $m\ddot{\underline{r}}$ = mass × acc in rotating frame

$$= -mg\,\hat{z} + m\omega^2\rho\,\hat{\rho}$$

$$= -\hat{z}\frac{\partial}{\partial z}(mgz) - \hat{\rho}\frac{\partial}{\partial \rho}\left(-\tfrac{1}{2}m\omega^2\rho^2\right)$$

$$= -\nabla\underbrace{\left(mgz - \tfrac{1}{2}m\omega^2\rho^2\right)}_{V_{eff}}$$

To have equilibrium:

$$V_{eff} = \text{const} = mgz - \tfrac{1}{2}m\omega^2\rho^2$$

or $z = \text{const.} + \frac{1}{2}m\omega^2\rho^2$: paraboloid

"Ground" frame:

in the liquid $\underbrace{-\overset{\text{density}}{d}\,\omega^2\rho\,\hat{\rho}}_{\text{acc. density}} = -\nabla p \underbrace{- dg\,\hat{z}}_{\text{downward grav}}$

so $p = p_0 - dgz + \frac{1}{2}d\omega^2\rho^2$. At the interface $p = p_{atm}$

$$\rightarrow z = z_0 + \frac{\omega^2}{2g}\rho^2$$