Non-inertial reference frames (Chapter 9)

Started this course by saying that "experiment" ⇒ there exist inertial ref frames where Newton's laws applied.

Obvious non-inertial frame = moving train

	train	ground
accelerates	you're pushed backwards	engine accelerates train, train pushes you
turns	you're pushed to side	train pushes you to provide centripetal force
	↑ extra force	↑ usual $F = ma$

Start in the inertial frame, change coordinate system to moving frame → find extra terms = "fictitious forces"

Linearly accelerating frames:

suppose $\underline{F} = m\ddot{\underline{r}}_0$ in inertial frame S_0

If frame S moves at velocity $\underline{V}$ + acceleration $\underline{A}$ wrt S_0,

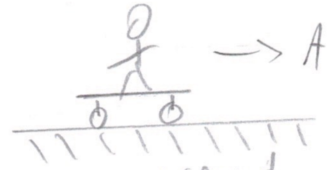
then $\underline{r}_0 = \underline{r} + \underline{\Delta}(t)$, $\dot{\underline{r}}_0 = \dot{\underline{r}} + \underline{V}$, $\ddot{\underline{r}}_0 = \ddot{\underline{r}} + \underline{A}$

↑ coord in S

→ $m\ddot{\underline{r}} = \underline{F} - m\underline{A}$ extra "inertial force" seen in S

In particular: if $\underline{A}$ = constant this is indistinguishable from gravity (at strength A) - form of "Principle of Equivalence"

Example 1 - moving train



ground frame: $A > 0$, $x_0 = \frac{1}{2} A t^2$

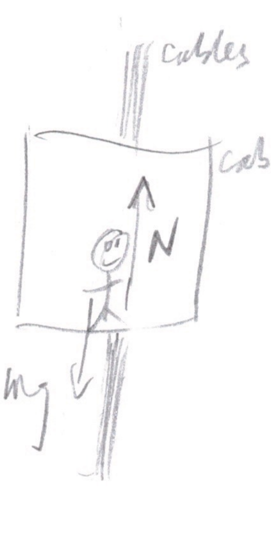
$F = mA$ = force applied to you to accelerate

train frame: $\ddot{x} = 0 = \ddot{x}_0 = F - mA$ ← inertial force

↑ applied force - friction from train floor

The inertial force pushes back + keeps you at rest in the car (pushes forward if decelerating)

Example 2 - elevator



ground frame $m\ddot{z}_0 = N - mg$

elevator frame $m\ddot{z} = N - mg - mA \rightarrow 0$

$A = \ddot{z}_0$

$= N - m(g + \ddot{z}_0)$

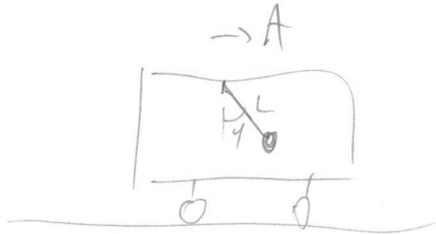
$\ddot{z}_0 = 0 \quad N = mg$

$\ddot{z}_0 > 0 \quad N = m(g + |\ddot{z}_0|)$ "extra g"

$\ddot{z}_0 < 0 \quad N = m(g - |\ddot{z}_0|)$ "less g"

free fall: $\ddot{z}_0 = -g \quad N = 0$

Ex 3 : Inertia – pendulum in moving train



$$m\ddot{\underline{r}}' = \underline{T} + m\underline{g} - m\underline{A} \qquad \text{train frame}$$

$$= \underline{T} + m\underline{g}_{eff} \qquad \underline{g}_{eff} = \underline{g} - \underline{A}$$

(1) looks like ordinary pendulum with g_{eff} at an angle, so oscillate about $\varphi = \alpha = \tan^{-1}\frac{A}{g}$ with period $\sqrt{\frac{g_{eff}}{L}}$

(2) Lagrangian method:

ground frame

$$\underline{r} = (L\sin\varphi + \Delta,\ L\cos\varphi)$$

$$\Delta = \int_0^t dt'\, V(t') = \int_0^t dt' \int_0^{t'} dt''\, A(t'')$$

$$\dot{\underline{r}} = (L\dot\varphi\cos\varphi + V,\ -L\dot\varphi\sin\varphi)$$

$$L = \frac{m}{2}\left[(L\dot\varphi\cos\varphi + V)^2 + (-L\dot\varphi\sin\varphi)^2\right] + mgL\cos\varphi$$

$$= \frac{m}{2}\left[L^2\dot\varphi^2 + 2L\dot\varphi V\cos\varphi + V^2\right] + mgL\cos\varphi$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\varphi} = \frac{\partial L}{\partial\varphi} \to \frac{d}{dt}\left(mL^2\dot\varphi + mLV\cos\varphi\right) =$$

$$= mL^2\ddot\varphi + mL\dot V\cos\varphi - \cancel{mLV\dot\varphi\sin\varphi}$$

$$= \cancel{-mL\dot\varphi V\sin\varphi} - mgL\sin\varphi$$

or $$mL^2\ddot{\varphi} = -mL\dot{V}\cos\varphi - mgL\sin\varphi$$

or $$\ddot{\varphi} = -\frac{g}{L}\sin\varphi - \frac{A}{L}\cos\varphi$$

usual pendulum

new term

for design $g = \cos\alpha \cdot g_{eff}$, $A = -\sin\alpha \cdot g_{eff}$

so $$\ddot{\varphi} = -\frac{g_{eff}}{L}\left(\sin\varphi\cos\alpha - \cos\varphi\sin\alpha\right)$$

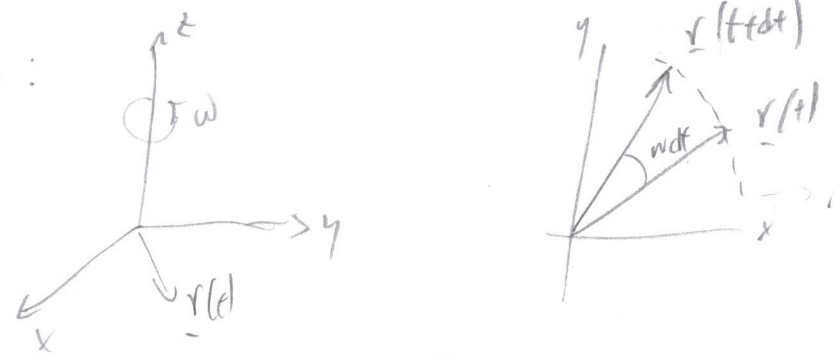
$$= -\frac{g_{eff}}{L}\sin\left(\varphi - \alpha\right)$$

: oscillates with $\omega = \sqrt{g_{eff}/L}$ about $\varphi = \alpha$!

— x —

Rotating reference frames:

reminder:



$$\underline{r} = (r\cos\varphi, r\sin\varphi, 0) \xrightarrow{dt} (r\cos(\varphi+\omega dt), \sin(\varphi+\omega dt), 0)$$

$$= r(\cos\varphi\cos\omega dt - \sin\varphi\sin\omega dt,$$
$$\sin\varphi\cos\omega dt + \cos\varphi\sin\omega dt, 0)$$

$$\simeq r(\cos\varphi - \omega dt\sin\varphi, \sin\varphi + \omega dt\cos\varphi, 0)$$

$$= \underline{r}(t) + d\underline{r}$$

Notice $\underline{\omega}\, dt \times \underline{r}(t) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega dt \\ r\cos\varphi & r\sin\varphi & 0 \end{vmatrix}$

$$= \hat{x}(-r\sin\varphi\cdot\omega dt) + \hat{y}(r\cos\varphi\cdot\omega dt)$$

$$= d\underline{r}$$

or $\dfrac{d\underline{r}}{dt} = \omega \times \underline{r}(t)$

If $\underline{Q} = Q_x\hat{x} + Q_y\hat{y} + Q_z\hat{z}$ = vector of fixed length $|\underline{Q}|$ which rotates at $\omega\hat{z}$, then Q_i = fixed but $\hat{x}$ & $\hat{y}$ rotate as $\underline{r}$ above

$$d\underline{Q}\big|_{rot} = Q_x\, \underline{\omega}\, dt \times \hat{x} + Q_y\, \underline{\omega}\, dt \times \hat{y} + 0$$

$$= \underline{\omega}\, dt \times \underline{Q}$$

This is completely general - only special assumption was to choose the $\hat{z}$ axis along $\underline{\omega}$ - always possible.

NB - $d\underline{Q}$ = change in $\underline{Q}$ due to rotation = change in the separate components in the old frame.

$\underline{Q}$ is still the vector which was fixed here.

In particular,

$$(\underline{Q} + d\underline{Q})^2 = \underline{Q}^2 + \underbrace{2\underline{Q}\cdot(\underline{\omega}\, dt \times \underline{Q})}_{0} + O(dt^2)$$

Now suppose $\underline{Q}$ is attached to some body frame S that rotates at $\underline{\omega}$ wrt the original neutral frame S_0, but allow $\underline{Q}$ to change in S. Then

$$d\underline{Q}\big|_{S_0} = d\underline{Q}\big|_S + d\underline{Q}\big|_{rot}$$

or

$$\frac{d\underline{Q}}{dt}\bigg|_{S_0} = \frac{d\underline{Q}}{dt}\bigg|_S + \underline{\omega} \times \underline{Q}$$

(or $\displaystyle \frac{d}{dt}\bigg|_{S_0} = \frac{d}{dt}\bigg|_S + \underline{\omega}\times$ for vectors)

Another way of saying this:

$$\underline{Q}(t) = \sum_{i=1}^{3} Q_i(t)\,\hat{e}_i(t) \qquad \hat{e}_i(t) = \text{unit vectors in } \underline{S}.$$

then

$$\left.\frac{d\underline{Q}}{dt}\right|_{S_0} = \sum_i \left(\dot{Q}_i(t)\,\hat{e}_i(t) + Q_i(t)\,\underbrace{\frac{d\hat{e}_i}{dt}}_{\underline{\omega}\times\hat{e}_i} \right)$$

$$= \left.\frac{d\underline{Q}}{dt}\right|_{S} + \underline{\omega}\times\underline{Q}.$$

NB: 1. there's only one $\underline{Q}$, not $\underline{Q}_S$ or $\underline{Q}_{S_0}$

2. $\left.\frac{d\omega}{dt}\right|_S = \left.\frac{d\omega}{dt}\right|_{S_0} + \underbrace{\omega\times\omega}_{0}$

so $\underline{\omega}$ = constant in one frame → constant in rotated frame.

+ the derivatives of $\underline{\omega}$ are the same as well.

Newton's eq in a rotating frame

$m\ddot{\underline{r}}\big|_{S_0} = \underline{F}$ in an inertial frame.

in S: $\dot{\underline{r}}\big|_{S_0} = \dot{\underline{r}}\big|_S + \underline{\omega}\times\underline{r}$ from the above.

$$\ddot{\underline{r}}\Big|_{S_0} = \left(\frac{d}{dt}\Big|_S + \underline{\omega}\times\right)\dot{\underline{r}}\Big|_{S_0}$$

$$= \left(\frac{d}{dt}\Big|_S + \underline{\omega}\times\right)\left(\dot{\underline{r}}\Big|_S + \underline{\omega}\times\underline{r}\right)$$

$$= \frac{d}{dt}\dot{\underline{r}}\Big|_S + \underline{\omega}\times\dot{\underline{r}}\Big|_S + \underbrace{\frac{d}{dt}\Big|_S(\underline{\omega}\times\underline{r})}_{\underline{\omega}\times\dot{\underline{r}}|_S + \dot{\underline{\omega}}\times\underline{r}} + \underline{\omega}\times(\underline{\omega}\times\underline{r})$$

only new $\underline{\omega}$

$$= \ddot{\underline{r}}\Big|_S + 2\underline{\omega}\times\dot{\underline{r}}\Big|_S + \dot{\underline{\omega}}\times\underline{r}\Big|_S + \underline{\omega}\times(\underline{\omega}\times\underline{r})$$

So, in the rotating frame (drop the S)

$$m\ddot{\underline{r}} = \underline{F} \underbrace{- 2m\underline{\omega}\times\dot{\underline{r}}}_{\text{Coriolis force}} \underbrace{- m\underline{\omega}\times(\underline{\omega}\times\underline{r})}_{\text{centrifugal force}} \underbrace{- m\dot{\underline{\omega}}\times\underline{r}}_{\text{azimuthal force}}$$

Centrifugal force:

first take $\underline{\omega} = \omega\hat{z}$, $\underline{r} \in$ x-y plane

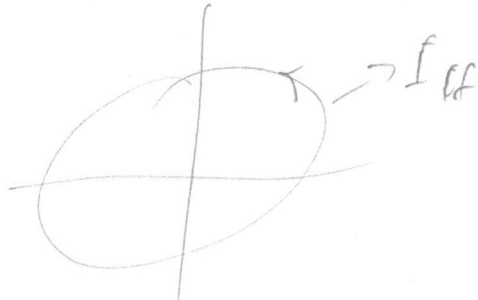
$$\underline{\omega}\times\underline{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & y & 0 \end{vmatrix} = -\hat{x}\,\omega y + \hat{y}\,\omega x$$

$$\underline{\omega}\times(\underline{\omega}\times\underline{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ -\omega y & \omega x & 0 \end{vmatrix}$$

$= \hat{x}(-\omega^2 x) + \hat{y}(-\omega^2 y) = -\omega^2 \underline{r}$

or $\underline{F}_{cf} = -m\underline{\omega}\times(\underline{\omega}\times\underline{r}) = +m\omega^2 r\,\hat{r}$ = outward force.

Physically, think of riding in a car in a circle



ground frame: you circle

acceleration = inward

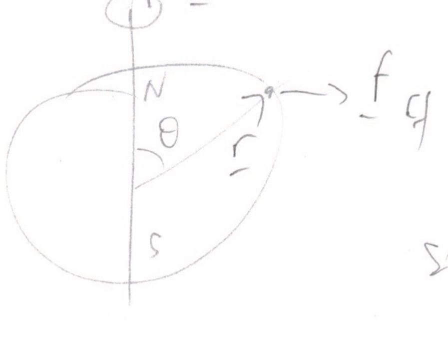
inward force from door of car

car frame: you're at rest but feel centrifugal

force pushing you against the door.

Same result (= force) on you.

Centrifugal force on Earth.



$\underline{\omega}\times\underline{r} = \omega R \sin\theta\,\hat{n}$

$\hat{n}$ = into the page

So $\underline{\omega}\times(\underline{\omega}\times\underline{r}) = \omega^2 R\sin\theta$ left

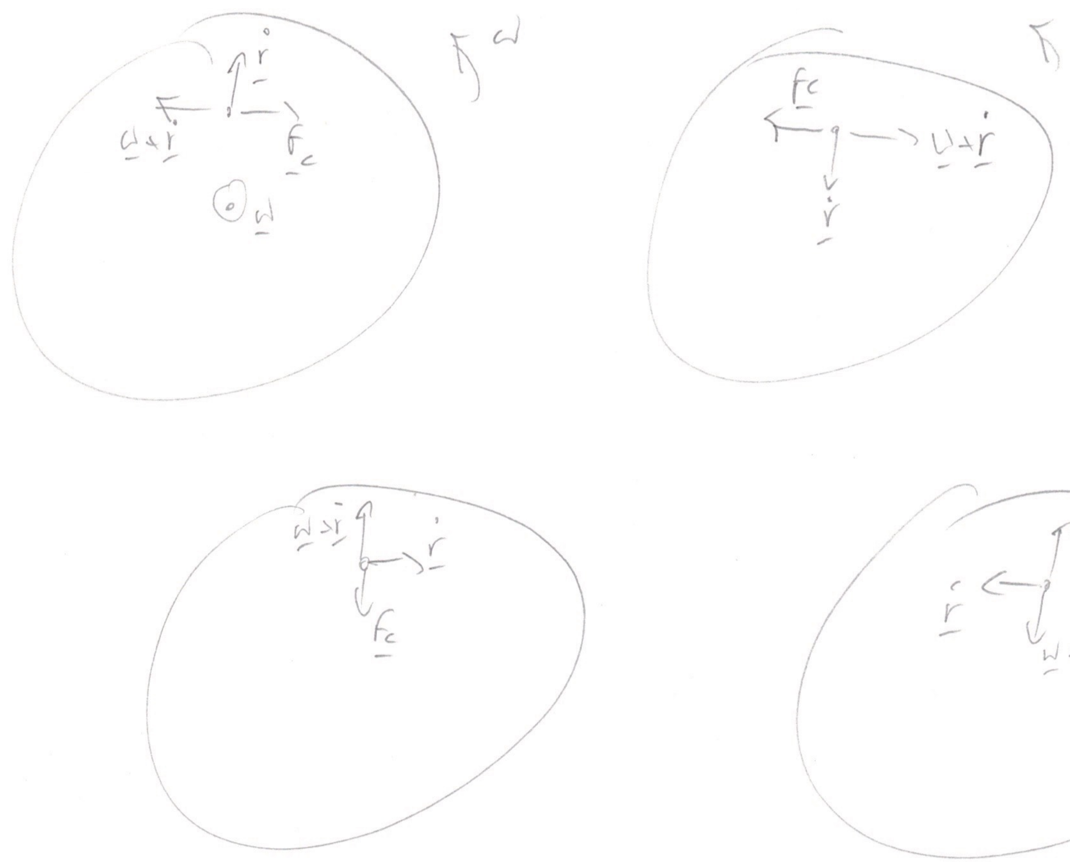
$-m\underline{\omega}\times(\underline{\omega}\times\underline{r}) = m\omega^2 R\sin\theta$ right

or $\underline{F}_{cf} = m\omega^2 R\sin\theta\,\hat{\rho}$.

NB $\omega_{Earth} = \frac{2\pi}{1\,day} = 7.3\times10^{-5}\,s^{-1}$, $R\sim6\times10^6\,m$, $\omega^2 R\sim10^{-2}\,m/s^2$

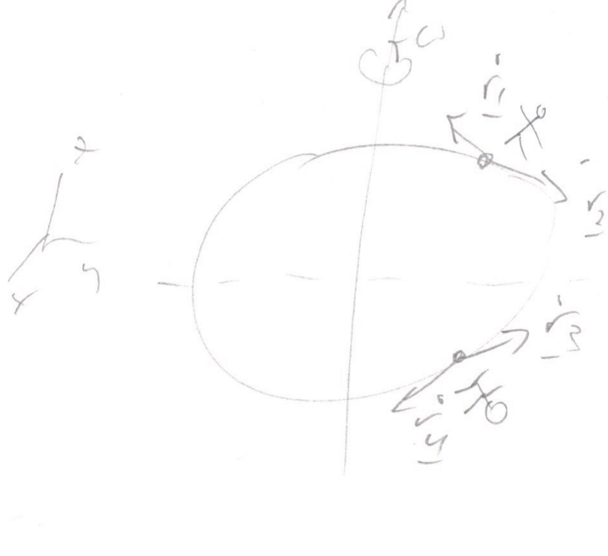
Coriolis force $\quad F_c = -2m\underline{\omega} \times \dot{\underline{r}}$

on a turntable:



force/deflection to right of direction of motion

on Earth



motion in xy plane the same
in Northern hemisphere

but in Southern you're "upside down"
+ right/left reversed
– deflection to left

or, direction of $\underline{\omega}$ reverses for motion N/S on Earth:

North hem: $\underline{\omega} \times \dot{\underline{r}}_1$ out of plane, $\underline{F}_{c,1}$ into plane
$\underline{\omega} \times \dot{\underline{r}}_2$ into plane $\underline{F}_{c,2}$ out of plane } deflect right

Southern hem: $\underline{\omega} \times \dot{\underline{r}}_3$ into plane, $\underline{F}_{c,3}$ out of plane
$\underline{\omega} \times \dot{\underline{r}}_4$ out of plane, $\underline{F}_{c,4}$ into plane } deflect left

→ Toilet bowl swirl story. *

Azimuthal force $\underline{f}_{az} = -m\dot{\underline{\omega}} \times \underline{r} = -m\dot{\omega}\, \hat{\omega} \times \underline{r}$

(not often considered)

accelerating turntable

$\underline{r}, \omega, \dot{\omega}$

extra force pushing back in rotating frame
↔ linear acc. case

* Note since $\omega \sim 10^{-4}\ \text{sec}^{-1}$ need large $\dot{r}$ to see an effect compared to $g = 10\ \text{m/s}^2$