Kepler orbits are closed curves: $r(\varphi+2\pi) = r(\varphi)$

Is this general?

No – look at $V(r) = -\frac{\gamma}{r} + \frac{\beta}{2r^2}$

= perturbation to usual V, "almost" a dipole correction

$V' = \frac{\gamma}{r^2} - \frac{\beta}{r^3} = \gamma u^2 - \beta u^3$

so $u'' + u = \frac{\mu}{l^2}(\gamma - \beta u)$

or $\frac{d^2}{d\varphi^2}\left[u - \frac{\mu\gamma/l^2}{1+\beta\mu/l^2}\right] = -\left(1+\frac{\beta\mu}{l^2}\right)\left[u - \frac{\mu\gamma/l^2}{1+\beta\mu/l^2}\right]$

$\rightarrow \quad u = \frac{\mu\gamma}{l^2+\beta\mu} + A\cos\left(\sqrt{1+\beta\mu/l^2}\,(\varphi-\delta)\right)$

so $u(\varphi+2\pi) \neq u(\varphi)$

— x —

Trajectory vs. time?

$l = \mu r^2\dot\varphi \rightarrow \frac{d\varphi}{dt} = \frac{l}{\mu^2}\cdot\frac{1}{c^2}(1+\varepsilon\cos\varphi)^2$

so $\frac{l}{\mu^2c^2}\,t = \int_{\varphi(0)}^{\varphi(t)} \frac{d\varphi}{(1+\varepsilon\cos\varphi)^2}$ = messy $f(\varphi)$

hard to invert for $\varphi(t)$; can't get $r(t) = \sqrt{l/\mu\dot\varphi(t)}$

Repulsive $\frac{1}{r}$ potential? $\quad V = \frac{\alpha}{r} \quad \alpha > 0$

$$u'' + u = \frac{\mu}{\ell^2 u^2} V'\left(\frac{1}{u}\right) = -\frac{\mu\alpha}{\ell^2} \equiv -\frac{1}{d}$$

so $\quad u = -\frac{1}{d} + A\cos\varphi \quad$ or $\quad r = \dfrac{d}{\varepsilon\cos\varphi - 1}$

Must have $\varepsilon \geq 1$ or else $r < 0$ for all φ, then $r \to \infty$ when $\cos\varphi \to 1/\varepsilon$

Plausible: V_{eff} is purely repulsive - no bound orbits

rewrite as $\quad \varepsilon x - r = d \quad \to (\varepsilon x - d)^2 = x^2 + y^2$

or $\ldots \quad \dfrac{(x-c)^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad$; hyperbola

$$c = \frac{d\varepsilon}{\varepsilon^2 - 1}, \quad a = \frac{d\varepsilon}{\varepsilon^2 - 1}, \quad b = \frac{d\varepsilon}{\sqrt{\varepsilon^2 - 1}}$$

For the energy note

$$r_{min} = \frac{d}{\varepsilon - 1} \quad \text{at } \varphi = 0$$

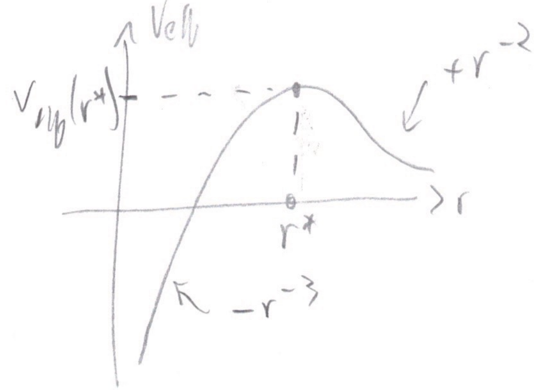
$$\dot{r}(r_{min}) = \left.\frac{d \sin\varphi\, \dot{\varphi}}{(\varepsilon\cos\varphi - 1)^2}\right|_{\varphi = 0} = 0$$

$$\Rightarrow E = V_{eff}(r_{min}) = \frac{\mu^2\alpha^2}{2\ell^2}(\varepsilon^2 - 1)$$

same result as for $V = \frac{-\alpha}{r}$

> 0 as expected.

Example: $V = -\frac{c}{3r^3}$ $\quad c > 0 \rightarrow \underline{F} = -\frac{c}{r^4}\hat{r}$



$$V_{eff} = \frac{l^2}{2\mu r^2} - \frac{c}{3r^3}$$

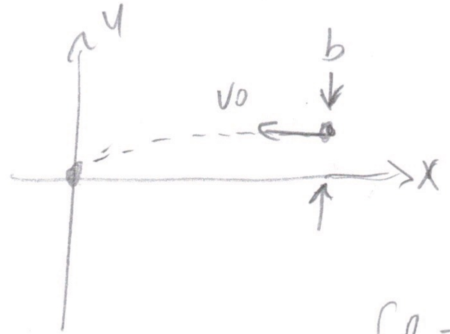
$$\left\{\begin{array}{l} V_{eff}' = \frac{-l^2}{\mu r^3} + \frac{c}{r^4} \rightarrow 0 \text{ at } r^* = \frac{c\mu}{l^2} \\ V_{eff}'' = \frac{3l^2}{\mu r^4} - \frac{4c}{r^5} \xrightarrow[r=r^*]{} \frac{-c}{r^{*5}} < 0 \\ V_{eff}(r^*) = \frac{l^6}{6\mu^3 c^2} \end{array}\right.$$

orbit eq: $u'' + u = \frac{\mu c}{l^2} u^2 = w \frac{dw}{du} + u$ $\quad$ if $w = \frac{du}{d\varphi}$

$$\rightarrow \frac{1}{2} w^2 = \frac{\mu c}{3l^2} u^3 - \frac{1}{2} u^2 = \frac{1}{2}\left(\frac{du}{d\varphi}\right)^2$$

$$\rightarrow \varphi = \frac{1}{\sqrt{2}} \int du \Big/ \sqrt{\left(\frac{\mu c}{3l^2} \cdots\right)} = \text{elliptic integral}$$

Further problem - m starts at $r = \infty$ with speed v_0 and impact parameter b. What's the largest b for which it gets to $r = 0$?



idea: if it gets past r^* with $\dot{r} < 0$ it gets sucked in by V_{eff}

so $b_{min} \leftrightarrow \dot{r} = 0$ at $r = r^*$

Here $\left\{\begin{array}{l} l = \mu v_0 b \\ E = \frac{1}{2} m v_0^2 = E(r^*) = V_{eff}(r^*) \cdots \end{array}\right.$

Since $l = \mu r^2 \dot{\varphi} = \text{const.}$, $\dot{\varphi} \rightarrow \infty$ as $r \rightarrow 0$

$\Rightarrow$ spirals inward

Virial Theorem (8.17) –

Define $G = \underline{r} \cdot \underline{p}$ = "virial" ; $\underline{r}$ + $\underline{p}$ = particle coord. + momentum

$$\frac{dG}{dt} = \dot{\underline{r}} \cdot \underline{p} + \underline{r} \cdot \dot{\underline{p}} = \underbrace{m\dot{\underline{r}}^2}_{=2T} + \underline{r} \cdot \underline{F}$$

Time-average: $\langle f(t) \rangle = \frac{1}{\tau} \int_0^\tau dt' f(t')$

$$\left\langle \frac{dG}{dt} \right\rangle = \frac{G(\tau) - G(0)}{\tau} = 2\langle T \rangle + \langle \underline{r} \cdot \underline{F} \rangle$$

For a closed Kepler orbit (or any confined orbit)

$\underline{r}$ + $\underline{p}$ are bounded → G is bounded

so $\frac{G(\tau) - G(0)}{\tau} \to 0$ for $\tau \to \infty$

If $V = kr^n$, $\underline{F} = -knr^{n-1}\hat{r}$ + $\underline{r} \cdot \underline{F} = -knr^n = -nV$

$\to \langle T \rangle = \frac{n}{2} \langle V \rangle$

Application (8.29) –

Half the sun disappears, $M_s \to \frac{1}{2} M_s$. Orbits?

Before: $E = T + V = \langle T \rangle + \langle V \rangle = \left(-\frac{1}{2}\langle V \rangle\right) + \langle V \rangle$

$= \frac{1}{2} \langle V \rangle$

After: $V \to \frac{1}{2} V$ and T unchanged, so $E \to 0$

→ new orbits are parabolas, solar system disintegrates

Normally one is given $V(r)$ & wants to know $r(\varphi)$ or $r(t)$

but can also ask what potential produces a given $r(\varphi)$?

$$\rightarrow \quad V'\left(\tfrac{1}{u}\right) = \frac{\ell^2 u^2}{\mu}\left[u''(\varphi) + u(\varphi)\right] = -F(u)$$

Spiral trajectories? take $r = ke^{\alpha\varphi}$, $\alpha = \begin{cases} \text{pos. outward} \\ \text{neg. inward} \end{cases}$

$$u'' + u = (\alpha^2+1)\,u$$

so $$F = -\frac{\ell^2}{\mu}(\alpha^2+1)\,u^3 = -\frac{\text{const}}{r^3} \qquad V = -\frac{\text{const}}{r^2}$$

trajectory as a fn of time?

$$\dot\varphi = \frac{\ell}{\mu}\, r^{-2} = \frac{\ell}{\mu k^2}\, e^{-2\alpha\varphi}$$

$$\rightarrow \quad e^{2\alpha\varphi}\, d\varphi = \frac{\ell}{\mu k^2}\, dt \quad \rightarrow \quad \frac{1}{2\alpha} e^{2\alpha\varphi} = \frac{\ell t}{\mu k^2} + C$$

or $$\varphi(t) = \frac{1}{2\alpha}\log\left(\frac{2\alpha\ell t}{\mu k^2} + C\right)$$

then $$r(t) = ke^{\alpha\varphi(t)} = k\sqrt{\frac{2\alpha\ell t}{\mu k^2} + C}$$

Note: 4 constants in solution k, α, ℓ, C

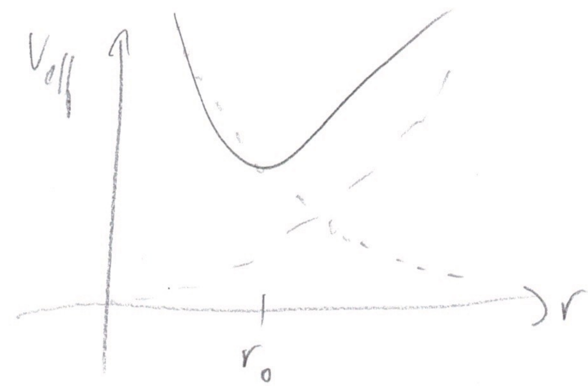
which is correct for one particle in 2d.

Also: sign $V \rightarrow$ inward/outward spirals

Example: Kepler + examples have $V \sim r^{-|n|}$

What about $V = \frac{1}{2}kr^2$? (8.13)

$$V_{eff} = V(r) + \frac{\ell^2}{2\mu r^2}$$



Possible circular orbit at r_0 where $V_{eff}' = 0$

$$\rightarrow \quad kr_0 - \frac{\ell^2}{\mu r_0^3} = 0 \qquad r_0 = \left(\ell^2/k\mu\right)^{1/4}$$

Stability?

$$V_{eff}'' = k + \frac{3\ell^2}{\mu r_0^4} = 4k > 0$$

1-d eq is $\mu r'' = -\frac{\partial V_{eff}}{\partial r}$

so if $r = r_0 + \delta r$: $\mu \delta \ddot{r} = -\frac{\partial^2 V_{eff}}{\partial r^2}\Big|_{r_0} \delta r$

$$= -4k\,\delta r$$

so $\delta r(t) \propto e^{\pm i\omega t}$ $\quad \omega = \sqrt{4k/\mu}$

Orbit Eq: $\quad u'' + u = \frac{\mu k}{\ell^2}\frac{1}{u^3}$ $\rightarrow$ numerical solution if $u \neq$ constant

(1) Stability of circular orbits?

(a) $F = -\frac{k}{r^n} \rightarrow V = \int^r dr' F(r') = -\frac{k}{n-1}\frac{1}{r^{n-1}} \cancel{+C}$

so $V_{eff}(r) = \frac{l^2}{2\mu r^2} - \frac{k}{n-1}\frac{1}{r^{n-1}}$

eq of motion is $\mu \ddot{r} = -\frac{\partial V_{eff}}{\partial r}$ so requirement is

$\frac{\partial V_{eff}}{\partial r} = 0$ & $\frac{\partial^2 V_{eff}}{\partial r^2} > 0$ at $r = r_0$

Here $V_{eff}' = -\frac{l^2}{\mu r^3} + \frac{k}{r^n} = 0$ at $r_0 = \left(\frac{k\mu}{l^2}\right)^{\frac{1}{n-3}}$

$V_{eff}'' = \frac{3l^2}{\mu r^4} - \frac{nk}{r^{n+1}}$

$= \frac{1}{r^4}\left(\frac{3l^2}{\mu} - \frac{nk}{r^{n-3}}\right)$

$\xrightarrow[r=r_0]{} \frac{1}{r_0^4}\left(\frac{3l^2}{\mu} - nk\cdot\left(\frac{l^2}{k\mu}\right)\right) = (3-n)\frac{l^2}{\mu r_0^4}$

so condition is $n < 3$.

(b) General case, force $F(r) = -\frac{\partial V}{\partial r}$

$V_{eff} = \frac{l^2}{2\mu r^2} + V(r)$

$$V_{eff}' = -\frac{\ell^2}{\mu r^3} - F(r) \to 0 \text{ at } r_0$$

$$V_{eff}'' = \frac{3\ell^2}{\mu r^4} - F'(r) \underset{r \to r_0}{\longrightarrow} -\frac{3}{r_0} F(r_0) - F'(r_0)$$

> 0 for a stable min

or (while $F(r_0) < 0$)

$$\frac{3}{r_0} + \frac{F'(r_0)}{F(r_0)} > 0$$

So if $F = -\frac{k}{r^n}$ $\quad F'/F = -n/r_0 \quad \to \quad 3-n > 0$

$F = -kr$ $\quad F'/F = 1/r_0 \quad$ always stable.

For explicit behavior of the perturbation,

let $r = r_0 + x$, $\quad V = V_{eff}(r_0) + \underset{0}{V_{eff}'(r_0)}\, x + \frac{1}{2} V_{eff}''(r_0)\, x^2$

$$\mu \ddot{x} = -V_{eff}''(r_0)\, x$$

$V_{eff}''(r_0) > 0 \to$ stable sinusoidal motion in x.

Changing orbits: suppose a satellite goes to a "bigger" orbit

$$r = \frac{c}{1+\varepsilon\cos\varphi}$$ so want c to increase

$$c = \frac{\ell^2}{\mu\gamma}$$ so want ℓ to increase.

Calculation is simplest at apogee/perigee where $\ell = \mu v r$ and a change in v directly changes ℓ (in practice may prefer other pt - harder calc.)

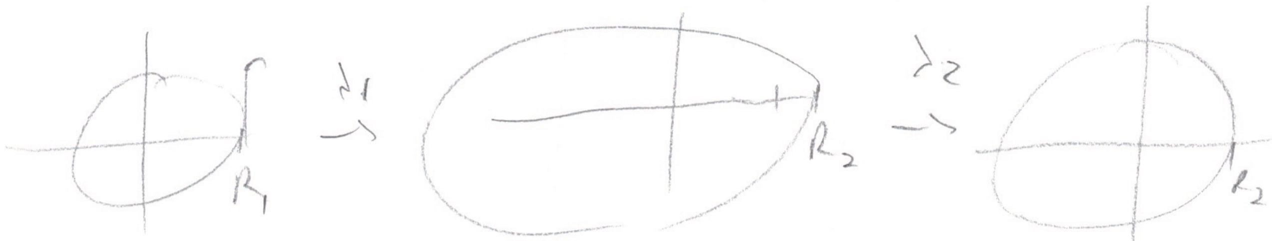
So if $v \to \lambda v$, $c \to \lambda^2 c$ $\quad r \to \dfrac{\lambda^2 c}{1+\varepsilon'\cos\varphi}$

where ε' is chosen so the orbits have a common point where the thrust is applied:

$$\frac{c}{1+\varepsilon} = \frac{\lambda^2 c}{1+\varepsilon'} \quad \text{or} \quad \varepsilon' = \lambda^2(1+\varepsilon) - 1$$

Could also contract orbit by $\lambda < 1$: backwards thrust.

Example (8.6 in Taylor): circular → circular at twice radius



$\varepsilon_1 = 0$ $\quad$ $\varepsilon_2' \neq 0$ $\quad$ $\varepsilon_3 = 0$

$c_1 = R_1$ $\quad$ $r_{min} = R_2$ $\quad$ $c_3 = R_3$

First orbit is $r = R_1$ or $c_1 = R_1$ & $\varepsilon_1 = 0$

Second orbit has $r_2 = R_3$ at apogee

$$r_2 = \frac{c_2}{1+\varepsilon_2 \cos\varphi} \xrightarrow{\text{apogee}} R_3 = \frac{c_2}{1-\varepsilon_2} = \frac{\lambda^2 R_1}{1-(\lambda^2-1)}$$

or $\lambda = \sqrt{2R_3/(R_1+R_3)}$

Third orbit is a circle of radius R_3 : $c_3 = R_3$ & $\varepsilon_3 = 0$

$$R_3 = \lambda'^2 c_2 = (\lambda\lambda')^2 R_1$$

or $\lambda' = \sqrt{R_1+R_3/2R_1}$

Alternate derivation of Kepler orbits:

define $\underline{A} = \underline{p} \times \underline{l} - \mu \gamma \hat{r}$ = "Runge-Lenz vector"

$$\frac{d\underline{A}}{dt} = \dot{\underline{p}} \times \underline{l} + \underbrace{\underline{p} \times \dot{\underline{l}}}_{=0} - \mu\gamma \frac{d\hat{r}}{dt}$$

$$\boxed{\underline{A} \times (\underline{B} \times \underline{C}) = \underline{B}(\underline{A} \cdot \underline{C}) - \underline{C}(\underline{A} \cdot \underline{B})}$$

$$\hookrightarrow \underline{F} \times \underline{l} = -\frac{\gamma \underline{r}}{r^3} \times (\underline{r} \times \mu \dot{\underline{r}})$$

$$= -\frac{\gamma\mu}{r^3}\left[\underline{r}(\underline{r} \cdot \dot{\underline{r}}) - \dot{\underline{r}}(r^2)\right]$$

$$= \gamma\mu\left[-\underline{r}\frac{\underline{r} \cdot \dot{\underline{r}}}{r^3} + \frac{1}{r}\dot{\underline{r}}\right]$$

$$= \gamma\mu \frac{d}{dt}(\underline{r}/r) \qquad \text{so } \frac{d\underline{A}}{dt} = 0$$

$$\underline{l} \cdot \underline{A} = \underline{l} \cdot (\underline{p} \times \underline{l}) - \frac{\mu\gamma}{r}(\underline{l} \cdot \underline{r}) = 0$$

So $\underline{A}$ = constant vector in plane $\perp \underline{l} \Rightarrow \underline{A} = A\hat{x}$

$$\underline{r} \cdot \underline{A} = \underline{r} \cdot (\underline{p} \times \underline{l}) - \mu\gamma r = (\underline{r} \times \underline{p}) \cdot \underline{l} - \mu\gamma r = l^2 - \mu\gamma r$$

$$= Ax = Ar\cos\varphi$$

$$\frac{1}{r} = \frac{\mu\gamma}{l^2}\Big(1 + \underbrace{\frac{A}{\mu\gamma}}_{\varepsilon}\cos\varphi\Big) \quad : \text{orbit eq}$$

Also $\underline{A}^2 = \cdots = 2\mu E l^2 + \mu^2\gamma^2$

So $\underset{E}{1} + \underset{\underline{l}}{3} + \underset{\underline{A}}{3} - 2 = 5$ conserved quantities

$$\underset{\underline{r},\,\underline{p}}{6 \text{ deg of freedom}} \longrightarrow 1 \text{ deg of freedom } (\varphi).$$