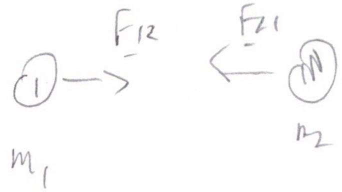
Central Forces + Orbits



$$\underline{F}_{12} = -\frac{\partial}{\partial \underline{r}_1} V\left(|\underline{r}_1 - \underline{r}_2|\right)$$

$$= -\frac{\underline{r}_1 - \underline{r}_2}{|\underline{r}_1 - \underline{r}_2|} V'\left(|\underline{r}_1 - \underline{r}_2|\right) = -\underline{F}_{21}$$

force along line joining centers

Goal: identify + characterize 2-particle trajectories

$\rightarrow$ $\left\{ \begin{array}{l} \text{"bound states"} \\ \text{"scattering states"} \end{array} \right\} \rightarrow$ orbits

Examples:

gravity $\quad V = -\dfrac{G m_1 m_2}{r}$ always attractive

Coulomb $\quad V = \dfrac{e_1 e_2}{4\pi\varepsilon_0 r}$ attractive/repulsive

Yukawa $\quad V = V_0 e^{-kr}/r$ nuclear, attractive

van der Waals $\quad V = -\dfrac{C}{r^6}$ atoms, attractive

$\quad \hookrightarrow$ polarization of electron clouds

CM coordinates:

$$\underline{R} = \frac{m_1 \underline{r}_1 + m_2 \underline{r}_2}{m_1 + m_2} \qquad \underline{r} = \underline{r}_1 - \underline{r}_2$$

$$\rightarrow \quad \underline{r}_1 = \underline{R} + \frac{m_2}{M} \underline{r} \qquad M = m_1 + m_2$$

$$\underline{r}_2 = \underline{R} - \frac{m_1}{M} \underline{r}$$

$$T = \tfrac{1}{2} m_1 \dot{\underline{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\underline{r}}_2^2$$

$$= \frac{m_1}{2}\left(\dot{\underline{R}} + \frac{m_2}{M}\dot{\underline{r}}\right)^2 + \frac{m_2}{2}\left(\dot{\underline{R}} - \frac{m_1}{M}\dot{\underline{r}}\right)^2$$

$$= \tfrac{1}{2} M \dot{\underline{R}}^2 + \frac{m_1 m_2}{2M}\dot{\underline{r}}^2 = T_{cm} + \tfrac{1}{2}\mu \dot{\underline{r}}^2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{or} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \quad \mu = \begin{cases} m/2 & m_1 = m_2 \\ m_1 & m_2 \gg m_1 \end{cases}$$

In latter case $\begin{cases} \underline{r}_2 \simeq \underline{R} \\ \underline{r}_1 \simeq \underline{R} + \underline{r} \end{cases}$ so almost a fixed force center
e.g. Sun & Earth
e^- & nucleus

$$L = \frac{M}{2}\dot{\underline{R}}^2 + \frac{\mu}{2}\dot{\underline{r}}^2 - V(\underline{r})$$

$$\frac{d}{dt}\left(M\dot{\underline{R}}\right) = 0 \quad \rightarrow \quad \dot{\underline{R}} = \text{const} = 0$$ by choice of ref. frame!

Notice $\underline{L} = \underline{r}_1 \times \underline{p}_1 + \underline{r}_2 \times \underline{p}_2$

$$= \left(\underline{R} + \frac{m_2}{M}\underline{r}\right) \times \underline{p}_1 + \left(\underline{R} - \frac{m_1}{M}\underline{r}\right) \times \underline{p}_2$$

$$= \underline{R} \times (\underline{p}_1 + \underline{p}_2) + \underline{r} \times \frac{m_1 m_2}{M}\left(\dot{\underline{r}}_1 - \dot{\underline{r}}_2\right)$$

$$= \underbrace{\underline{R} \times M\dot{\underline{R}}}_{\underline{L}_{cm}} + \underbrace{\underline{r} \times \mu \dot{\underline{r}}}_{\underline{L}'}$$

& also $\frac{dL}{dt} = \underline{r}_1 \times \underset{\underset{\underline{F}_{12}}{\shortparallel}}{\dot{\underline{p}}_1} + \underline{r}_2 \times \underset{\underset{\underline{F}_{21} = -\underline{F}_{12}}{\shortparallel}}{\dot{\underline{p}}_2} = (\underline{r}_1 - \underline{r}_2) \times \underline{F}_{12}$

$= 0$ for central forces.

So (1) Choose ref frame where $\dot{\underline{R}} = 0 \Rightarrow \underline{L}_{cm} = 0$

$\rightarrow \underline{L}' = $ const

(2) The direction of $\underline{r} \times \dot{\underline{r}}$ is $\hat{\underline{L}}' = $ const so

the plane of $\underline{r}$ & $\dot{\underline{r}}$ is fixed in space $\rightarrow$ xy plane

Take polar coord in xy-plane $\rightarrow$

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r) \quad \text{effectively}$$

$$\underline{L}' = \underline{r} \times (\mu\dot{\underline{r}}) = \underline{r} \times \mu(\dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}) = \mu r^2\dot{\varphi}\hat{z}$$
$$\equiv l\hat{z}$$

with $l = $ const

Other conserved quantity - energy

(1) L indep of t $\Rightarrow$

$$H = \sum_i p_i \dot{q}_i - L = E = \text{const}$$

$$E = (M\dot{\underline{R}}) \cdot \dot{\underline{R}} - (\mu\dot{\underline{r}}) \cdot \dot{\underline{r}} - L$$

$$E = \frac{1}{2} M \dot{\underline{R}}^2 + \frac{1}{2}\mu \dot{\underline{r}}^2 + V(r) \rightarrow \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\varphi}^2 + V(r)$$

$$\underbrace{}_{\text{ignore}} = \frac{1}{2}\mu \dot{r}^2 + \left(\frac{\ell^2}{2\mu r^2} + V(r)\right) \equiv \frac{1}{2}\mu \dot{r}^2 + V_{eff}(r)$$

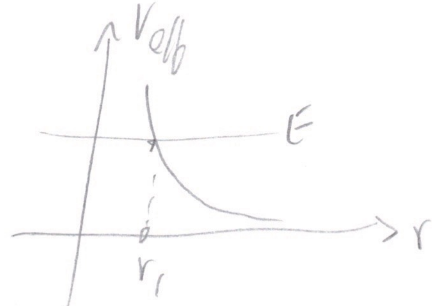
(2) $$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2} M \dot{\underline{R}}^2 + \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)\right]$$

$$= 0 + \mu \dot{r}\ddot{r} - \frac{\ell^2 \dot{r}}{\mu r^3} + V'(r)\dot{r} = \dot{r}\left(\mu \ddot{r} - \frac{\ell^2}{\mu r^3} + V'(r)\right)$$

$$= 0$$

The term $\frac{\ell^2}{\mu r^2}$ in V_{eff} is the "centrifugal barrier"

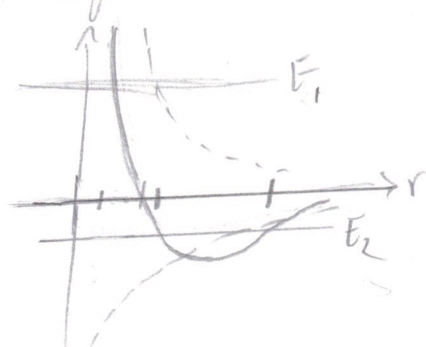
e.g. if $V(r) > 0$



at r_1 where $V(r_1) = E$, $\dot{r} = 0$ and motion must be in $r > r_1$

if $V(r) < 0$ it depends

e.g.



$V = -\frac{\gamma}{r}$

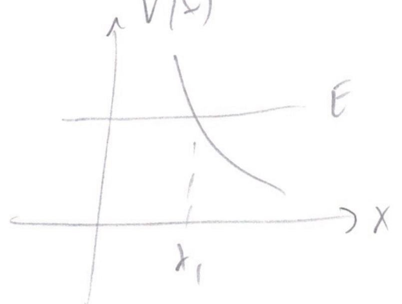
$E_1 > 0$: $r > r_1$

$E_2 < 0$: $r \in$

but if $V = \frac{-\gamma}{r^4}$, $\frac{\ell^2}{2\mu r^2}$ is negligible for small r

Note distinction between 1-d + 2d:

1-d

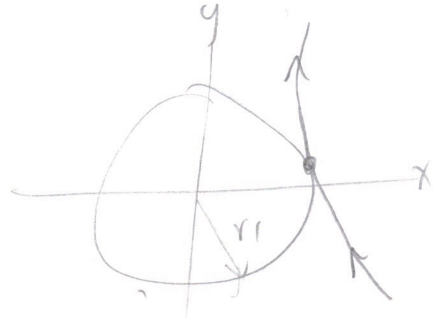


$E = \frac{1}{2} m \dot{x}^2 + V(x)$

when $E = V(x_1)$ $\dot{x} = 0$

particle stops + reverses

2-d



$E = \frac{1}{2} \mu \dot{r}^2 + V_{eff}(r)$

when $E = V_{eff}(r_1)$ $\dot{r} = 0$

but $\dot{\varphi} \neq 0$ (if $l \neq 0$)

so $\dot{\underline{r}} \neq 0$

Shape of the orbit:

in principle use same method as in 1-d

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - V_{eff}(r)\right)} \xrightarrow[\text{integrate}]{} r(t)$$

then $$\frac{d\varphi}{dt} = l/\mu r^2(t) \xrightarrow[\text{integrate}]{} \varphi(t)$$

$\xrightarrow[\text{eliminate } t]{} r(\varphi)$

but 2-step method is clumsy & integrals might be tough

Better: $$\mu\ddot{r} = \frac{l^2}{\mu r^3} - V'(r)$$

write $$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi} = \frac{l}{\mu r^2}\frac{d}{d\varphi}$$

(NB: $l = \mu r^2\dot{\varphi}$ = constant $\rightarrow$ sign of $\dot{\varphi}$ const. $\rightarrow$ φ monotonic in t)

then let $u \equiv \frac{1}{r} = u(\varphi)$

$$\dot{r} = \frac{dr}{dt} = \frac{lu^2}{\mu}\frac{d}{d\varphi}\left(\frac{1}{u}\right) = -\frac{l}{\mu}u'(\varphi)$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{lu^2}{\mu}\frac{d}{d\varphi}\left(-\frac{l}{\mu}u'(\varphi)\right) = -\frac{l^2u^2}{\mu^2}u''(\varphi)$$

so $$-\frac{l^2}{\mu^2}u^2u''(\varphi) = -V_{eff}'\left(\frac{1}{u}\right) = \frac{l^2u^3}{\mu} - V'\left(\frac{1}{u}\right)$$

or $$\boxed{u'' + u = \frac{\mu}{l^2u^2}V'\left(\frac{1}{u}\right)}$$ "orbit equation" for $u(\varphi)$

Get back to $r(t), \varphi(t)$?

given $r(\varphi)$, $\dot\varphi = \frac{\ell^2}{\mu} u^2(\varphi) \longrightarrow \varphi(t)$, $r(t) = r(\varphi(t))$

Trivial example: no force, $V' = 0$

here $u'' + u = 0 \to u = A\cos(\varphi - \delta) = \frac{1}{r}$

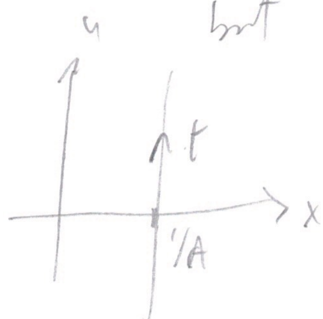
choose coordinates so that δ = phase = 0 (rot. about $\hat z$)

$\to r\cos\varphi = x = \frac{1}{A}$ = constant

Then $\ell = \mu r^2 \dot\varphi = \mu\left(\frac{1}{A}\sec^2\varphi\right)^2\dot\varphi = \frac{\mu}{A^2}\frac{d}{dt}(\tan\varphi)$

so $\tan\varphi = \tan\varphi_0 + \frac{A^2\ell}{\mu} t$

but $\tan\varphi = y/x = Ay \to y$ = linear in t



moves along the line $x = \frac{1}{A}$

Gravitational case - Kepler orbits

$$V = -\frac{\gamma}{r} \rightarrow V' = \frac{\gamma}{r^2} = \gamma u^2$$

orbit eq $\rightarrow$ $u'' + u = \frac{\mu}{l^2 u^2} V'(u) \rightarrow \frac{\gamma \mu}{l^2} \equiv \frac{1}{c}$

to solve : $\frac{d^2}{d\varphi^2}\left(u - \frac{1}{c}\right) = -\left(u - \frac{1}{c}\right)$

so $u = \frac{1}{c} + A\cos\varphi$ (choose phase $= 0$ again)

let $A = \frac{\varepsilon}{c}$ so $\frac{1}{r} = \frac{1}{c}\left(1 + \varepsilon\cos\varphi\right)$

or $$\boxed{r(\varphi) = \frac{c}{1 + \varepsilon\cos\varphi}}$$

c = constant determined by $\dot{\varphi}$ + r at $t=0$

ε = other constant $\rightarrow$ type of orbit

notice $\varepsilon \rightarrow -\varepsilon$ is the same as $\varphi \rightarrow \varphi + \pi$

which is a trivial relabeling of φ

$\rightarrow$ choose $\varepsilon \geq 0$ + look at different values

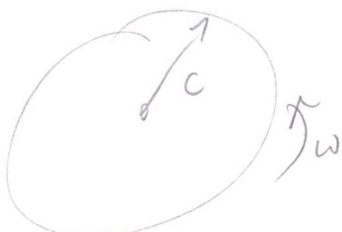
$\boxed{\varepsilon = 0}$ $r = c$ - circular orbit

$$l = \mu r^2 \dot{\theta} = \text{const} \quad \text{so} \quad \dot{\theta} = \omega = \text{const}$$

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} - \frac{\gamma}{r} \rightarrow \frac{l^2}{2\mu c^2} - \frac{1}{c}\left(\frac{l^2}{\mu c}\right)$$

$$= -\frac{l^2}{2\mu c^2} = -\frac{1}{2}\mu c^2 \omega^2$$

direct calculation:



$$R \rightarrow a_{\text{centripetal}} = \mu \omega^2 c = \frac{\gamma}{c^2}$$

$$E = \frac{1}{2}\mu(\omega c)^2 - \frac{\gamma}{c} = -\frac{1}{2}\mu c^2 \omega^2$$

stability? $\dfrac{d^2 V_{\text{eff}}}{dr^2} = \dfrac{3l^2}{\mu r^4} - \dfrac{2\gamma}{r^3} \xrightarrow[\gamma = l^2/\mu c]{r=c} + \dfrac{l^2}{\mu c^4} > 0$

so $r = c$ is a minimum of V' $\rightarrow$ stable

$\boxed{0 < \varepsilon < 1}$: $r = \dfrac{c}{1 + \varepsilon \cos\varphi}$ is finite, $\dfrac{c}{1-\varepsilon} \le r \le \dfrac{c}{1+\varepsilon}$

actually an ellipse: $r + \varepsilon r \cos\varphi = \sqrt{x^2 + y^2} + \varepsilon x = c$

or $x^2 + y^2 = (c - \varepsilon x)^2$

$$(1 - \varepsilon^2)x^2 + 2\varepsilon c x + y^2 = c^2$$

or $(1-\varepsilon^2)\Big[x + \underbrace{\frac{\varepsilon c}{1-\varepsilon^2}}_{\equiv d}\Big]^2 - \frac{\varepsilon^2 c^2}{1-\varepsilon^2} + y^2 = c^2$

or $(x+d)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{c^2}{1-\varepsilon^2} + \frac{\varepsilon^2 c^2}{(1-\varepsilon^2)^2} = \frac{c^2}{(1-\varepsilon^2)^2}$

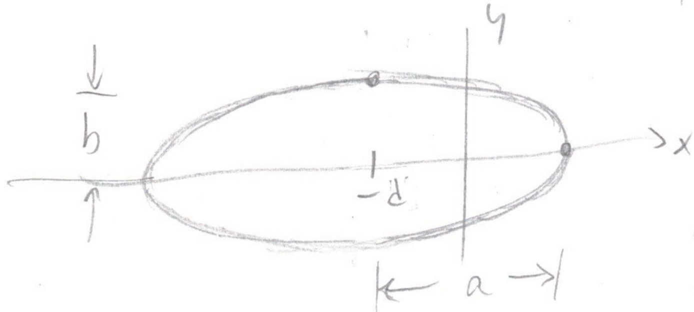
or $\frac{(x+d)^2}{c^2/(1-\varepsilon^2)^2} + \frac{y^2}{c^2/(1-\varepsilon^2)} = 1$

= ellipse centred at $(-d, 0)$ axes $\frac{c}{1-\varepsilon^2}$ & $\frac{c}{\sqrt{1-\varepsilon^2}}$

Ratio of major/minor axes $= \frac{1}{\sqrt{1-\varepsilon^2}} \equiv \frac{a}{b}$

ε = "eccentricity" $\leftrightarrow$ $\varepsilon = 0$ $b = a$

$\varepsilon \to 1$ $\frac{a}{b} \to \infty$



Period: If A = area swept by radius vector

$$\frac{dA}{dt} = \frac{1}{2}|\underline{r} \times \dot{\underline{r}}| = \frac{l}{2\mu}$$

& $\tau = \frac{A}{dA/dt} = \frac{\pi a b}{l/2\mu} = \ldots$

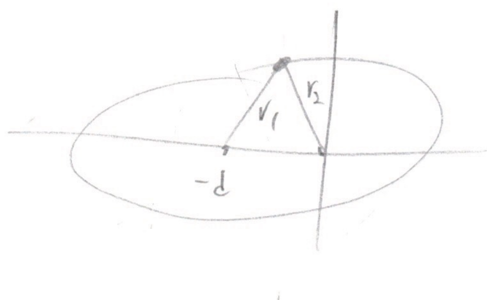
see below

$$T^2 \cong \frac{4\pi^2}{GM_s} a^3 \qquad (\text{approx is } M \simeq M_s)$$

which is Kepler #3.

Also Kepler #1: orbit is an ellipse with the Sun at a focus

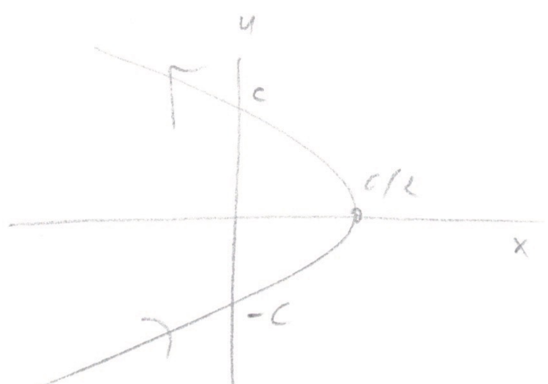
$r_1 + r_2 =$ constant



→ boring geometry exercise.

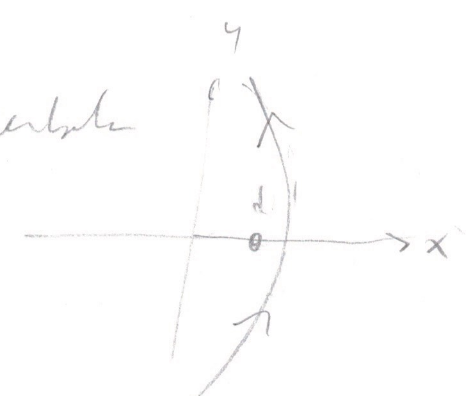
$\varepsilon = 1$: $\quad (1-\varepsilon^2)x^2 + 2\varepsilon c x + y^2 = c^2$

$$\xrightarrow[\varepsilon \to 1]{} \quad x = \frac{c}{2} - \frac{y^2}{2c} \quad \text{: parabola}$$

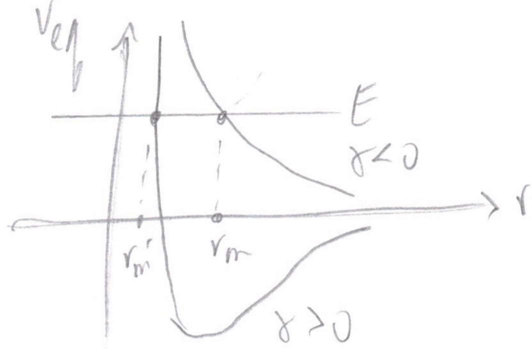


$\varepsilon > 1$: see formula as $\varepsilon < 1$ – hyperbola

$$\frac{(x+d)^2}{c^2/(\varepsilon^2-1)^2} - \frac{y^2}{c^2/(\varepsilon^2-1)} = 1$$



Energy vs ε:



$$V_{eff} = \frac{l^2}{2\mu r^2} - \frac{\gamma}{r}$$

$\rightarrow$ At r_m (or r_m') $E = V_{eff}(r_m) \rightarrow T=0 \rightarrow \dot{r}=0$

from $r = \dfrac{c}{1+\varepsilon\cos\varphi}$, $r_m = \dfrac{c}{1+\varepsilon}$ at $\varphi=0$

$$\dot{r} = \frac{c\dot{\varphi}}{(1+\varepsilon\cos\varphi)^2}\sin\varphi \quad \text{so } \dot{r}=0 \text{ at } r_m$$

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r) \underset{\substack{r=r_m \\ c=l^2/2\mu}}{\longrightarrow} 0 + \frac{l^2}{2\mu}\left(\frac{\gamma\mu}{l^2}(1+\varepsilon)\right)^2 - \gamma\,\frac{\gamma\mu}{l^2}(1+\varepsilon)$$

$$= \frac{\gamma^2\mu}{2l^2}(\varepsilon^2-1) = \begin{cases} \text{positive for } \varepsilon>1 \\ 0 \text{ for } \varepsilon<0 \\ \text{negative } 0<\varepsilon<1 \end{cases}$$

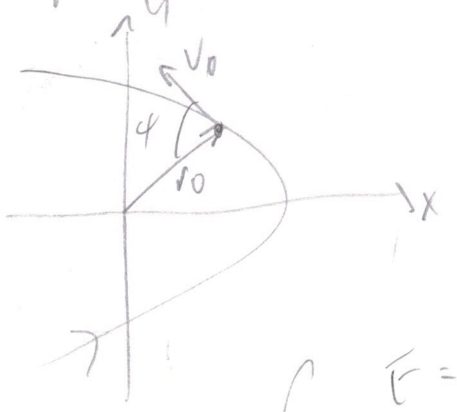
$E>0$ $\rightarrow$ particle can escape to ∞

$E=0$ " " (barely)

$E<0$ particle is bound - can't escape

Harder derivation: substitute for $r, \dot{r}, \dot{\varphi}$ in E at any φ ...

Example: A comet has speed v_0 at distance r_0 from the Sun, where the velocity vector makes an angle ψ with the radius vector. $\varepsilon = ?$



at r_0

$$E = \frac{1}{2}\mu v_0^2 - \frac{\gamma}{r_0} > 0 \quad \text{for a comet}$$

$$\mu \triangleq m_{comet} \, , \quad \gamma = G m_{comet} M_{Sun}$$

$$l = |\underline{r} \times \mu \underline{v}| = r_0 \mu v_0 \sin\psi$$

From $E(\varepsilon)$, invert

$$\rightarrow \varepsilon = \sqrt{\frac{2l^2 E}{\gamma^2 \mu} + 1} > 1$$