More about constraints

previously: to get to the generalized coordinates eliminate constrained variables by hand:

eliminate x + y using $x = l\sin\varphi$ $y = l\cos\varphi$

$$T = \frac{m}{2}\left(\dot x^2 + \dot y^2\right) \to \frac{ml^2}{2}\dot\varphi^2$$

$x = x(z)$, $y = y(z)$

$$T = \frac{m}{2}\left(\dot x^2 + \dot y^2 + \dot z^2\right) \to \frac{m}{2}\left(x'(z)^2 + y'(z)^2 + 1\right)\dot z^2$$

But what if the transformation eq is transcendental

e.g. $x\sin x = ze^{-z}$ or $x + \alpha\cos(\beta y) = z$ or ...

→ Can't get the Lagrangian in closed form, let alone the eqs. of motion.

Resolution: go back to calculus, recall method of "Lagrange multipliers", extend to Euler-Lagrange

Lagrange multipliers:

find the min/max of $\varphi(x,y)$ subject to $f(x,y)=0$.

(1) solve $f=0$ for $y=y(x)$ + substitute into φ:

$\varphi(x, y(x)) = \Phi(x)$, then min/max of Φ

$$\rightarrow \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\Big|_{y=y(x)} y'(x) = 0 \rightarrow \text{solution } x^*, y^* = y(x^*)$$

(2) If not possible:

$$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = 0 \quad \text{at } (x^*, y^*) \text{ for extremum}$$

if x + y were independent then $\varphi_x = \varphi_y = 0$

– usual condition for $\varphi(x,y)$

here, $f=0$ so $df = f_x\, dx + f_y\, dy = 0$

$$\rightarrow dy = -\frac{f_x}{f_y} dx \rightarrow d\varphi = \left(\varphi_x - \varphi_y \frac{f_x}{f_y}\right) dx$$

so $d\varphi = 0 \rightarrow \underbrace{\varphi_x - \varphi_y \frac{f_x}{f_y} = 0 \text{ and } f=0}_{\text{solve for } (x^*, y^*)}$

note $\frac{dy}{dx} = -\frac{f_x}{f_y}$: same thing

(3) Extremize $F(x,y,\lambda) = \varphi(x,y) + \lambda f(x,y)$ wrt x, y + λ

$$\rightarrow \left.\begin{aligned} \frac{\partial F}{\partial x} &= 0 = \varphi_x + \lambda f_x \\ \frac{\partial F}{\partial y} &= 0 = \varphi_y + \lambda f_y \end{aligned}\right\} \quad \varphi_x - \varphi_y \frac{f_x}{f_y} = 0$$

$$\frac{\partial F}{\partial \lambda} = 0 = f$$

same thing

Example: extrema of $\varphi(x,y) = (x+1)^2 + 2y$, subject to $x = 3y$

direct = substitute $\Phi(y) = \varphi(x(y), y) = (3y+1)^2 + 2y$

$= 9y^2 + 8y + 1$

$\Phi'(y) = 18y + 8 = 0$ at $y^* = -\frac{4}{9}$, $x^* = -\frac{4}{3}$

Lagrange mult.

$F(x,y,\lambda) = (x+1)^2 + 2y + \lambda(x-3y) = \varphi + \lambda f$

$F_x = 0 = 2(x+1) + \lambda \qquad \rightarrow x = -\frac{4}{3}$

$F_y = 0 = 2 - 3\lambda \rightarrow \lambda = \frac{2}{3}$

$F_\lambda = 0 = x - 3y \qquad \rightarrow y = -\frac{4}{9}$

— x —

Recast method (3):

$d\varphi = \varphi_x \, dx + \varphi_y \, dy = 0$

x + y are not independent, can't say $\varphi_x = \varphi_y = 0$

but $f = 0$ so $df = f_x \, dx + f_y \, dy = 0$

Add: $d\varphi + \lambda \, df = 0$ for any λ

or $(\varphi_x + \lambda f_x) \, dx + (\varphi_y + \lambda f_y) \, dy = 0$

choose λ such that $\varphi_y + \lambda f_y = 0$; $\lambda = -\varphi_y / f_y$

$\rightarrow \left(\varphi_x - \varphi_y \frac{f_x}{f_y}\right) dx = 0$ for any dx

$\rightarrow \varphi_x - \varphi_y \, {f_x}/{f_y} = 0$.

Back to Calculus of Variations:

extremize $I[f,g] = \int_{x_1}^{x_2} dx\, \varphi(f(x), f'(x), g(x), g'(x), x)$

subject to $\psi(f(x), g(x)) = 0$ - this is the constraint

write $f(x) = f_0(x) + \alpha\eta(x) \quad g(x) = g_1(x) + \beta\xi(x)$

α, β = small; η, ξ = arbitrary but $=0$ at x_1, x_2

$$I[f,g] = I[f_0, g_1] + \int_{x_1}^{x_2} dx \left\{ \alpha\eta(x)\left[\frac{\partial\varphi}{\partial f} - \frac{d}{dx}\frac{\partial\varphi}{\partial f_x}\right] + \beta\xi(x)\left[\frac{\partial\varphi}{\partial g} - \frac{d}{dx}\frac{\partial\varphi}{\partial g_x}\right]\right\} + \dots$$

Cannot say coefficients of η & $\xi = 0$ because f & g cannot be varied independently

But $\lambda\delta\psi = \lambda\left(\frac{\partial\psi}{\partial f}\delta f + \frac{\partial\psi}{\partial g}\delta g\right) = \lambda\left(\frac{\partial\psi}{\partial f}\alpha\eta + \frac{\partial\psi}{\partial g}\beta\xi\right) = 0$

for any $\lambda(x)$

So $\delta I = \delta I + \lambda\delta\psi$

$$= \int_{x_1}^{x_2} dx \left\{ \alpha\left[\frac{\partial\varphi}{\partial f} - \frac{d}{dx}\frac{\partial\varphi}{\partial f_x} + \lambda\frac{\partial\psi}{\partial f}\right]\eta(x) + \beta\left[\frac{\partial\varphi}{\partial g} - \frac{d}{dx}\frac{\partial\varphi}{\partial g_x} + \lambda\frac{\partial\psi}{\partial g}\right]\xi(x)\right\} = 0$$

Choose λ such that $[\ ]_2 = 0$, then vary $\eta(x)$ at will

$\to [\ ]_1 = 0$

So $\frac{\partial \varphi}{\partial f} - \frac{d}{dx}\frac{\partial \varphi}{\partial f_x} + \lambda \frac{\partial \psi}{\partial f} = 0$, $\frac{\partial \varphi}{\partial g} - \frac{d}{dx}\frac{\partial \varphi}{\partial g_x} + \lambda \frac{\partial \psi}{\partial g} = 0$, $\psi = 0$

are 3 eqs. for $f(x), g(x), \lambda(x)$ at the extremum

This is equivalent to extremizing

$$J[f,g,\lambda] = \int dx \left[\varphi(f,g,\dots) + \lambda \psi(f,g) \right]$$

with respect to f, g and λ.

Caution: last version ↑ wrong if ψ involves f_x or g_x

<u>Example</u>

masses m + M connected by a massless inextensible string

m moves in the x-y plane

M moves vertically

$$L = \frac{m}{2}\left(\dot r^2 + r^2 \dot\theta^2\right) + \frac{M}{2}\dot z^2 - Mgz$$

constraint: $r + z = l = $ const. or $\psi = r + z - l = 0$

Direct method: $L \to \frac{M+m}{2}\dot r^2 + \frac{m}{2} r^2 \dot\theta^2 + Mgr + \text{constant}$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r} = (m+M)\ddot r - mr^2\dot\theta^2 - Mg = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = -\frac{d}{dt}\left(mr^2\dot\theta\right) = 0$$

multiplier method:

$$\tilde{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{M}{2}\dot{z}^2 - Mgz + \lambda\left(r+z-l\right)$$

r: $\quad m\ddot{r} - mr\dot{\theta}^2 - \lambda = 0$

θ: $\quad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \quad$ → same thing

z: $\quad M\ddot{z} + Mg - \lambda = 0 \quad \rightarrow \lambda = M(\ddot{z}+g)$

1^{st} eq → $\underbrace{(m+M)\ddot{r} - mr\dot{\theta}^2}_{\lambda} - Mg = 0 \quad$ ∴ same eq.

Other example: mass m moves on the surface $f(\underline{r}) = 0$ in 3d

$$\tilde{L} = \frac{1}{2}m\dot{\underline{r}}^2 - V(\underline{r}) + \lambda f(\underline{r})$$

$$\rightarrow m\ddot{\underline{r}} = -\frac{\partial V}{\partial \underline{r}} + \lambda\frac{\partial f}{\partial \underline{r}}$$ — what is this?

$f(\underline{r}) = 0 = f(\underline{r}+d\underline{r}) = f(\underline{r}) + d\underline{r}\cdot\frac{\partial f}{\partial \underline{r}} + \cdots$

so for $d\underline{r} \rightarrow 0$, $\quad d\underline{r}\cdot\frac{\partial f}{\partial \underline{r}} = 0$,

but when $d\underline{r} \rightarrow 0$, $d\underline{r}$ → tangent to surface

∴ $\partial f/\partial \underline{r}$ is $\perp$ to any tangent to $f=0$

→ $\partial f/\partial \underline{r} \propto \hat{n}$ = normal vector, $\hat{n} = \frac{\partial f}{\partial \underline{r}} \Big/ \left|\frac{\partial f}{\partial \underline{r}}\right|$

$$m\ddot{\underline{r}} = -\frac{\partial V}{\partial \underline{r}} + \hat{n}\lambda\left|\frac{\partial f}{\partial \underline{r}}\right|$$

∴ $\lambda\left|\frac{\partial f}{\partial \underline{r}}\right|$ = normal force holding m on the surface

General case: $L = L(\{q_i(t)\}, \{\dot{q}_i(t)\}, t)$, $i = 1, 2, \cdots, n$

subject to $\psi_\alpha(\{q_i(t)\}) = 0$, $\alpha = 1, 2, \cdots k$ constraints

$$\delta \int L\,dt = 0 \rightarrow \int dt \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i = 0$$

without constraints all δq_i are independent

$\rightarrow$ n usual E-L eqs.

with constraints $\psi_\alpha = 0 \rightarrow \sum_i \frac{\partial \psi_\alpha}{\partial q_i} \delta q_i = 0$

or $\sum_\alpha \lambda_\alpha(t) \sum_i \frac{\partial \psi_\alpha}{\partial q_i} \delta q_i = 0$

Add;

$$0 = \int dt \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_\alpha \lambda_\alpha(t) \frac{\partial \psi_\alpha}{\partial q_i} \right] \delta q_i = 0$$

Choose λ_α such that the first k [ ]'s = 0,

n-m left, multiplied by n-m independent δq_i

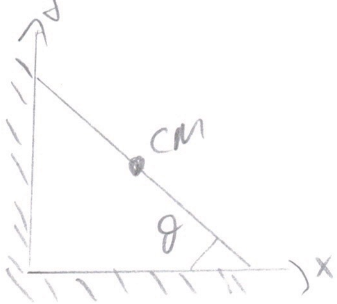
$$\rightarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_\alpha \lambda_\alpha \frac{\partial \psi_\alpha}{\partial q_i} = 0 \quad i = 1, 2, \cdots n$$

$+ \quad \psi_\alpha = 0 \quad \alpha = 1, 2, \cdots k$

n+k eqs for n q_i and k λ_α

Snappy version:

$$\delta \int dt \left(L + \sum_\alpha \lambda_\alpha \psi_\alpha \right) = 0$$



Let x = position of left end $= x_c - L\cos\theta$

y = " " bottom end $= y_c - L\sin\theta$

+ treat $x=0$ & $y=0$ as constraints

$$L = \frac{m}{2}\left((\dot{x} - L\sin\theta\,\dot{\theta})^2 + (\dot{y} + L\cos\theta\,\dot{\theta})^2\right) + \frac{1}{2}I\dot{\theta}^2 - mg(y + L\sin\theta) + \lambda_x(x) + \lambda_y(y)$$

$$= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + 2L\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta)\right) + \frac{1}{2}(mL^2 + I)\dot{\theta}^2 - mg(y + L\sin\theta) + \lambda_x x + \lambda_y y$$

x: $\frac{d}{dt}\left(m\dot{x} - mL\dot{\theta}\sin\theta\right) = \lambda_x$

y: $\frac{d}{dt}\left(m\dot{y} + mL\dot{\theta}\cos\theta\right) = \lambda_y - mg$

θ: $\frac{d}{dt}\left((mL^2 + I)\dot{\theta} + mL(\dot{y}\cos\theta - \dot{x}\sin\theta)\right) = -mgL\cos\theta$

$\lambda_{x,y}$: $x = y = 0$

Impose constraints:

$$-mL(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) = \lambda_x$$
$$mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = \lambda_y - mg$$
$$(I + mL^2)\ddot{\theta} = -mgL\cos\theta$$

Also have conservation of energy ($L \neq L(t)$ $H = E$)

$$mgL\sin\theta_0 = mgL\sin\theta + \frac{1}{2}(I + mL^2)\dot{\theta}^2$$

↑ start at rest at θ_0

Substitute for $\ddot{\theta}$ + $\dot{\theta}^2$ in $\lambda_{x,y}$:

$$\lambda_x = -mL\left[-\frac{mgL\cos\theta}{I+mL^2}\cdot\sin\theta + \frac{2mgL(\sin\theta_0-\sin\theta)}{I+mL^2}\cdot\cos\theta\right]$$

$$= \frac{m^2gL}{I+mL^2}\cos\theta\,(2\sin\theta_0 - 3\sin\theta)$$

$\rightarrow 0$ when $\sin\theta = \frac{2}{3}\sin\theta_0$

or height $= \frac{2}{3}\times$ original height

Is the bottom still in contact?

$$\lambda_y = mg + mL\left[-\frac{mgL\cos\theta}{I+mL^2}\cdot\cos\theta - \frac{2mgL(\sin\theta_0-\sin\theta)}{I+mL^2}\cdot\sin\theta\right]$$

$$\xrightarrow{I=\frac{1}{3}mL^2} \frac{mg}{4}\left[1+9\sin^2\theta - 6\sin\theta_0\sin\theta\right]$$

$$\lambda_y(\text{start}) = \frac{mg}{4}(1+3\sin^2\theta_0)$$

$$\lambda_y(\lambda_x=0) = \frac{mg}{4}$$

so λ_x keeps its sign: bottom wall force $\neq 0$

$\rightarrow$ ladder stays in contact w/ the bottom until left end detaches

$\lambda_y = 0$? $\sin\theta = \frac{1}{3}\left(\sin\theta_0 \pm \sqrt{\sin^2\theta_0 - 1}\right)$: never.

Noether's Theorem - formal treatment of symmetry

Suppose $\{q_i(t),\ i=1,2,\cdots\}$ solves Lagrange's Eqs.

look at $q_i \to q_i(t) + \delta q_i(t)$

$\dot{q}_i \to \dot{q}_i + \frac{d}{dt}\delta q_i$

$$S = \int dt\, L(\underline{q}, \underline{\dot{q}}, t)$$

$$\to \int dt\, \underbrace{L(\underline{q}+\delta\underline{q}, \underline{\dot{q}}+\delta\underline{\dot{q}}, t)}$$

$$= L(\underline{q}, \underline{\dot{q}}, t) + \sum_i \underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i}\delta q_i + \sum_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i}\frac{d}{dt}\delta q_i$$

$$= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

$$= L + \frac{d}{dt}\sum_i p_i \delta q_i$$

$$= S + \sum_i p_i \delta q_i \Big|_{t_1}^{t_2}$$

So: if S is invariant under $\underline{q} \to \underline{q} + \delta\underline{q}$, $Q \equiv \sum_i p_i \delta q_i = \text{constant}$

e.g. if $L = \sum_i \frac{1}{2} m_i \dot{\underline{r}}_i^2 - \sum_{i<j} V_2(\underline{r}_i - \underline{r}_j)$

$\underline{r}_i \to \underline{r}_i + \underline{\varepsilon}$ leaves L + therefore S unchanged

$\Rightarrow Q = \sum_i \underline{p}_i \cdot \underline{\varepsilon} = \underline{P} \cdot \underline{\varepsilon} = \text{const}$

$\to$ conservation of momentum

if furthermore $V_2 = V_2(|\underline{r}_i - \underline{r}_j|)$ = central force

look at $\underline{r}_i \to \underline{r}_i + \underline{\delta\theta} \times \underline{r}_i$ = infinitesimal rotation by $\underline{\delta\theta}$

$$\dot{\underline{r}}_i \to \dot{\underline{r}}_i + \underline{\delta\theta} \times \dot{\underline{r}}_i$$

then $$\dot{\underline{r}}_i^2 \to \dot{\underline{r}}_i^2 + 2\underbrace{\dot{\underline{r}}_i \cdot (\underline{\delta\theta} \times \dot{\underline{r}}_i)}_{=0} + O(\delta\theta^2)$$

$$(\underline{r}_i - \underline{r}_j)^2 \to \left((\underline{r}_i - \underline{r}_j) + \underline{\delta\theta} \times (\underline{r}_i - \underline{r}_j)\right)^2$$
$$= (\underline{r}_i - \underline{r}_j)^2 + 2\underbrace{(\underline{r}_i - \underline{r}_j) \cdot \underline{\delta\theta} \times (\underline{r}_i - \underline{r}_j)}_{=0} + O(\delta\theta^2)$$

so L + S are unchanged.

Here $$Q = \sum_i \underline{p}_i \cdot (\underline{\delta\theta} \times \underline{r}_i) = \sum (\underline{\delta\theta} \times \underline{r}_i) \cdot \underline{p}_i$$
$$= \sum_i \underline{\delta\theta} \cdot (\underline{r}_i \times \underline{p}_i) = \underline{\delta\theta} \times \sum_i \underline{r}_i \times \underline{p}_i$$
$$= \text{constant}$$

+ since $\underline{\delta\theta}$ has any direction $\sum_i \underline{r}_i \times \underline{p}_i = \underline{L} = \text{const.}$

→ Conservation of $\underline{L}$.

In (quantum) field theory, L involves different types of particle in symmetry classes, + Noether's Thm is used to identify conserved quantities.