Calculus of Variations:

Finding extrema (max/min etc.) of a function is standard, but what about the extrema of a functional.

$f(x)$: assign a real # to any x

$I[f]$: " " to any function $f(x)$

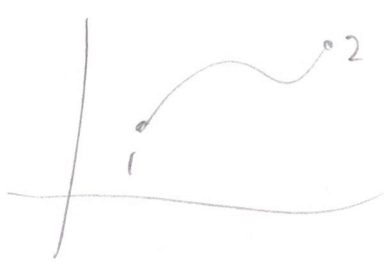
e.g. $I = \max\limits_{a<x<b} f(x)$ or $\frac{1}{L}\int_0^L dx\, f^2(x) = \langle f^2 \rangle$ or

$I = \int_a^b dx\, \varphi\big(f(x), f'(x), x\big)$ interesting case

↳ ordinary fn, e.g. polynomial, $\sqrt{\ }$, ...

Why is this interesting?

1. Shortest distance btw 2 points



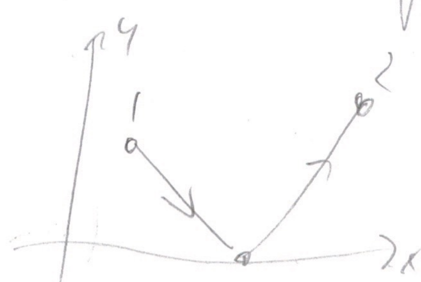
$ds = \sqrt{dx^2 + dy^2}$

$L = \int_1^2 ds = \int_{x_1}^{x_2} dx \sqrt{1 + (dy/dx)^2}$

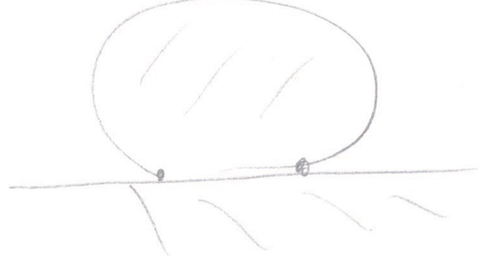
min $y(x) = ?$

"Obvious" in the plane, but on some curved surface?

1a. What about shortest dist curve which hits the axis?



2. Liquid drop on a solid



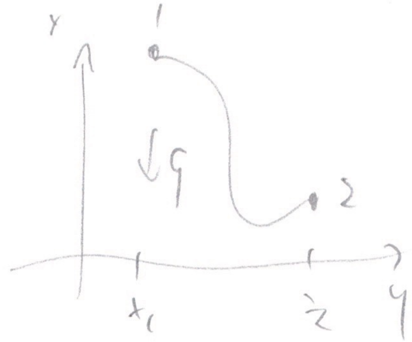
$E = \int_S$ surface tension of drop ↑ shape dependent

$+$ int. with solid ← contact pts. dependent

$+$ gravity

Equilibrium shape?

3. Brachistochrone problem – min time for a particle to fall under gravity btw 2 pts



$$T = \int \frac{ds}{v} = \int \frac{\sqrt{dx^2+dy^2}}{v(y)}$$

$$\rightarrow \int_{x_1}^{x_2} dx \, \frac{\sqrt{1+(dy/dx)^2}}{\sqrt{2g\,y(x)}} = \int_{y_1}^{y_1} dy \sqrt{\frac{1+x'(y)}{2gy}}$$

4. Travel time for light in a medium

$$T = \int \frac{ds}{c/n(x,y)} = \frac{1}{c}\int dx \, n(x,y)\sqrt{1+(dy/dx)^2}$$

5. Mechanics: extremum of $S = \int_{t_1}^{t_2} dt \left[T(\underline{r},\dot{\underline{r}}) - V(\underline{r}) \right]$

the "action"

→ Lagrange's Eq.

In calculus, $y(x)$ has a minimum at x_0 if, for small α,

$$y(x_0 \pm \alpha) > y(x_0) \quad \text{or} \quad y(x_0) \pm \alpha y'(x_0) > y(x_0)$$

which is only possible for + and − cases if $y'(x_0) = 0$.

Likewise for a maximum: $y(x_0) > y(x_0 \pm \alpha)$ ...

So if $I[f]$ is to be min or max at $f_0(x)$, require that

$I[f_0 + \delta f] > I[f_0]$ for any "sign" of (small) δf

any shape here

$\to \delta f = \alpha \eta(x)$, α = small parameter, $\eta(x)$ = any function

To make the problem well-defined specify $f(x_1) = f_1$ & $f(x_2) = f_2$ say

which requires $\eta = 0$ at x_1 & x_2.

$\to I[f_0 + \alpha\eta] = I[f_0] + \alpha I_1[\eta] + O(\alpha^2)$ so require $I_1 = 0$

$$\hookrightarrow \int_{x_1}^{x_2} dx\, \underbrace{\varphi(f_0 + \alpha\eta, f_0' + \alpha\eta', x)}_{} $$

$$= \varphi(f_0, f_0') + \alpha\eta \frac{\partial\varphi}{\partial f}\Big|_0 + \alpha\eta' \frac{\partial\varphi}{\partial f'}\Big|_0 + \dots$$

$$I[f_0 + \alpha\eta] = I[f_0] + \alpha \int_{x_1}^{x_2} dx \left(\eta(x) \frac{\partial\varphi}{\partial f}\Big|_0 + \underbrace{\eta' \frac{\partial\varphi}{\partial f'}\Big|_0}_{} + \dots \right)$$

$\Downarrow$ ibp

$$\eta \frac{\partial\varphi}{\partial f'}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} dx\, \eta \frac{d}{dx}\left(\frac{\partial\varphi}{\partial f'}\right)$$

$$= I[f_0] + \alpha \int_{x_1}^{x_2} dx\, \eta(x) \underbrace{\left[\frac{\partial\varphi}{\partial f} - \frac{d}{dx}\frac{\partial\varphi}{\partial f'} \right]}_{\text{must } = 0\ !} + \dots$$

So: the function $f(x)$ which extremizes (max or min)

$$I[f] = \int_{x_1}^{x_2} dx\, \varphi\left(f(x), f'(x), x\right)$$

subject to $f(x_1) = f_1$ + $f(x_2) = f_2$ satisfies

$$\frac{\partial \varphi}{\partial f} - \frac{d}{dx}\frac{\partial \varphi}{\partial f'} = 0 \quad : \text{Euler - Lagrange Eqs}$$

Application to mechanics:

$$x \to t, \quad f(x) \to q(t) \quad \varphi \to L(q, \dot{q}, t) = T - V$$

extremizing $\int_{t_1}^{t_2} dt\, L(q, \dot{q}, t)$ with $q(t_{1,2}) = q_{1,2}$

gives $$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \quad : \text{Lagrange eqs.}$$

$\Rightarrow$ Basic principle of classical mechanics is

" $S = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t)$ is an extremum."

(Realistically, need restrictions to systems described by V.)

Generalizes to 2d, 3d, more than one $q(t)$, continuous fields, quantum mechanical systems, ...

called Hamilton's Principle (sometimes)

Example: $L = \int_{x_1}^{x_2} dx \sqrt{1+y'(x)^2}$ = length of line btw 1 & 2

$y(x_i) = y_i \quad i = 1,2$

$$\frac{d}{dx}\left(\frac{y'(x)}{\sqrt{1+y'^2}}\right) = 0 \rightarrow \frac{y'}{\sqrt{1+y'^2}} = \text{const} \rightarrow y' = \text{const}$$

so $y(x)$ = linear fn btw the 2 points.

max or min?

$$\sqrt{1+y'^2} = \sqrt{1+(y_0'+\delta y')^2} = \sqrt{1+y_0'^2 + 2y_0'\delta y' + \delta y'^2}$$

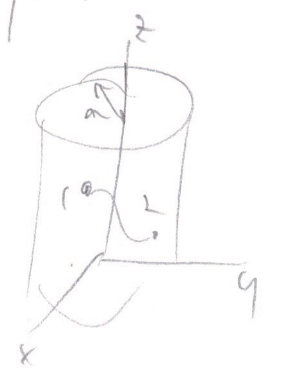
$$= \sqrt{1+y_0'^2}\left[1 + \frac{1}{2}\cdot\frac{2y_0'\delta y' + \delta y'^2}{1+y_0'^2} - \frac{1}{8}\left(\frac{2y_0'\delta y' + \delta y'^2}{1+y_0'^2}\right)^2 + \dots\right]$$

$$= \underbrace{\sqrt{1+y_0'^2}}_{\text{length of the straight line}} + \underbrace{\cancel{\delta y' \frac{2y_0'}{\sqrt{1+y_0'^2}}}}_{0 \text{ ibp}} + \delta y'^2 \left[\underbrace{\frac{1/2}{\sqrt{1+y_0'^2}} - \frac{1}{2}\frac{y_0'^2}{(1+y_0'^2)^{3/2}}}_{\frac{1}{2}(1+y_0'^2)^{-3/2} > 0}\right]$$

Ex 2: geodesics

shortest dist btw 2 points on a surface

cylinder



pick $\underline{r}_1 = (a, 0, 0)$ on x axis, xz plane

$\underline{r}_2 = (a\cos\varphi_2, a\sin\varphi_2, z_2)$

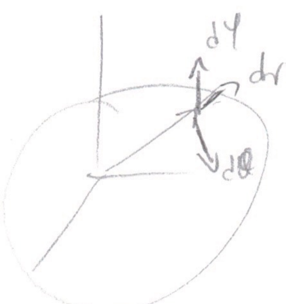
$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$
$$= a^2 d\varphi^2 + dz^2$$

$$L = \int_1^2 ds = \int_0^{\varphi_2} d\varphi \sqrt{a^2 + (dz/d\varphi)^2}$$

same problem $\rightarrow$ $z'(\varphi) = \text{const}$ $\rightarrow$ $z = \frac{z_2}{\varphi_2}\varphi$

Similarly for a sphere.

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$



$\underline{r}_1$ = North pole $\theta = \varphi = 0$

$\underline{r}_2$ = $\{\theta = \theta_2, \ \varphi = 0$ by choice of axes$\}$

$$L = \int_1^2 \sqrt{R^2 d\theta^2 + R^2 \sin^2\theta \, d\varphi^2}$$

$$= R \int_0^{\theta_2} d\theta \underbrace{\sqrt{1 + \sin^2\theta \, \varphi'(\theta)^2}}_{\Phi(\varphi', \theta)}$$

so $$\frac{d}{d\theta}\frac{\partial \Phi}{\partial \varphi'} = \frac{\partial \Phi}{\partial \varphi} = 0$$

$$\rightarrow \frac{\sin^2\theta \, \varphi'(\theta)}{\sqrt{1 + \sin^2\theta \, \varphi'^2}} = C$$

$\theta = \varphi = 0$ on the curve $\therefore$ $C = 0$ $\Rightarrow$ $\varphi' = 0$

$\therefore$ φ = constant $\therefore$ segment of great circle through North pole

Brachistochrone Problem

$$T = \frac{1}{\sqrt{2g}} \int_1^2 \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}}$$

Method 1 (Goldstein) – write the solution as $x(y)$

so $$\sqrt{2g}\, T = \int_{y_1}^{y_2} dy \underbrace{\sqrt{\frac{1 + x'(y)^2}{y}}}_{\varphi(x(y), x'(y); y)}$$

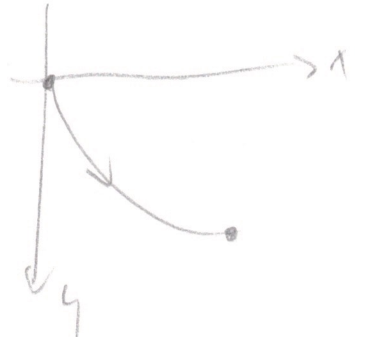
$$\frac{d}{dy} \underbrace{\frac{\partial \varphi}{\partial x'(y)}}_{} = \frac{\partial \varphi}{\partial x(y)} = 0 \quad \text{again}$$

$$\frac{1}{\sqrt{y}} \frac{x'}{\sqrt{1 + x'^2}} = C = \frac{1}{\sqrt{2a}} \qquad 2a\, x'^2 = y(1 + x'^2)$$

so $$x(y) = \int_0^y dy \sqrt{\frac{y}{2a - y}}$$

$$\xrightarrow[y = a(1-\cos\theta)]{} \int a \sin\theta\, d\theta \sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)} \cdot \frac{1-\cos\theta}{1-\cos\theta}}$$

$$x = \int a(1 - \cos\theta)\, d\theta = a(\theta - \sin\theta)$$



$$\left.\begin{aligned} x &= a\theta^3/3 + \dots \\ y &= a\theta^2/2 + \dots \end{aligned}\right\} \; y = x^{3/2} + \dots$$

method 2 — look for solution as $y(x)$

$$\sqrt{2g}\, T = \int_{x_1}^{x_2} dx \sqrt{\frac{1+y'(x)^2}{y(x)}} \Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{y(1+y'^2)}}\right) = -\frac{\sqrt{1+y'^2}}{2y^{3/2}(x)}$$

: hopelessly complicated!

But: $\varphi(y, y', x)$ is actually independent of x

let $\quad \psi \equiv y' \dfrac{\partial \varphi}{\partial y'} - \varphi \qquad$ [cf: $H = p\dot{q} - L$]

$$\frac{d\psi}{dx} = \cancel{y'' \frac{\partial \varphi}{\partial y'}} + y' \underbrace{\frac{d}{dx}\left(\frac{\partial \varphi}{\partial y'}\right)}_{d\varphi/\partial y} - \frac{\partial \varphi}{\partial y} y' - \cancel{\frac{\partial \varphi}{\partial y'} y''}$$

$= 0$

so $\psi(y, y') =$ constant in x

$$\rightarrow \quad y' \cdot \frac{y'}{\sqrt{y(1+y'^2)}} - \sqrt{\frac{1+y'^2}{y}} = \frac{y'^2 - (1+y'^2)}{\sqrt{y(1+y'^2)}} = \frac{1}{\sqrt{y(1+y'^2)}} = 2a$$

convenient choice

or $\quad y' = \sqrt{\dfrac{2a-y}{y}}$

$$x(y) = \int dy \sqrt{\frac{y}{2a-y}} = \text{previous result.}$$

6.22

String of length l starts & ends on the x-axis, curve which has max area?

$A = \int_0^{x_1} dx\, y(x)$ but this is the wrong form for C of V $-$ $x_1 = ?$

Instead, use arclength as dependent variable $ds = \sqrt{dx^2+dy^2}$

so $dx = \sqrt{dx^2 - dy^2} = \sqrt{ds^2 - (y'(s)\,ds)^2} = ds\,\sqrt{1-y'(s)^2}$

and $A = \int_0^l ds\, \underbrace{y(s)\sqrt{1-y'(s)^2}}_{\varphi(y,y')}$

$\varphi(y,y')$ indep of s $\rightarrow$ $y'\frac{\partial\varphi}{\partial y'} - \varphi = C$

or $y' \cdot y \frac{-y'}{\sqrt{1-y'^2}} - y\sqrt{1-y'^2} = \frac{-y}{\sqrt{1-y'^2}} = C$

$\rightarrow$ $y' = \sqrt{1-y^2/c^2}$ $\rightarrow$ $s = \int_0^y \frac{dy}{\sqrt{1-y^2/c^2}} = C\sin^{-1} y/c$

so $y = C\sin s/c$ B.C.: $y(0)=y(l)=0 \Rightarrow C = \frac{l}{n\pi}$

$n = 1, 2, \ldots$

Then $x'(s) = \sqrt{1-y'(s)^2} = \sqrt{1-\cos^2 n\pi s/l} = \sin\frac{n\pi s}{l}$

so $x(s) = \frac{l}{n\pi}\left(1-\cos\frac{n\pi s}{c}\right)$, $y(s) = \frac{l}{n\pi}\sin\frac{n\pi l}{c}$

What's n? $\left(x-\frac{l}{n\pi}\right)^2 + y^2 = \left(l/n\pi\right)^2$: circle of radius $\frac{l}{n\pi}$

so $A = \max$ when $n=1$.

More than one unknown function:

$$I[f,g] = \int dx\, \varphi(f(x), g(x), f'(x), g'(x), x)$$

let $f(x) = f_0(x) + \alpha\eta(x)$ $\quad g(x) = g_0(x) + \beta\xi(x)$

where f_0, g_0 = minimizers, α, β = independent small #s

η, ξ = " arbitrary fns.

[i.e., look at arbitrary independent variations in f & g]

$$\varphi = \varphi(f_0, g_0) + \frac{\partial\varphi}{\partial f}\cdot\alpha\eta + \frac{\partial\varphi}{\partial f'}\cdot\alpha\eta' + \frac{\partial\varphi}{\partial g}\cdot\beta\xi + \frac{\partial\varphi}{\partial g'}\cdot\beta\xi' + \cdots$$

$$I = I[f_0, g_0] + \alpha\int dx\, \eta(x)\left[\frac{\partial\varphi}{\partial f} - \frac{d}{dx}\frac{\partial\varphi}{\partial f'}\right] + \beta\int dx\, \xi(x)\left[\frac{\partial\varphi}{\partial g} - \frac{d}{dx}\frac{\partial\varphi}{\partial g'}\right] + \cdots$$

For an extremum, require $\delta I = 0$ to first order in α and β, for any $\eta(x)$ & $\xi(x)$

$$\rightarrow \frac{\partial\varphi}{\partial f} - \frac{d}{dx}\frac{\partial\varphi}{\partial f'} = 0 \quad \text{and} \quad \frac{\partial\varphi}{\partial g} - \frac{d}{dx}\frac{\partial\varphi}{\partial g'} = 0$$

General case: $I[f_1, f_2, \cdots] = \int dx\, \varphi(f_1, f_1', f_2, f_2', f_3 \cdots)$

$$\rightarrow \frac{\partial\varphi}{\partial f_i} - \frac{d}{dx}\frac{\partial\varphi}{\partial f_i'} = 0 \qquad i = 1, 2, \cdots$$

More than one dependent variable (relevant for fields)

$$I[f] = \int_R dx\,dy\, \varphi\left(f, \underset{\overset{\shortparallel}{f_x}}{\frac{\partial f}{\partial x}}, \underset{\overset{\shortparallel}{f_y}}{\frac{\partial f}{\partial y}}, x, y\right), \quad f \text{ given on } \partial R$$

If $f_0(x,y)$ = minimizing function, look at variation

$f = f_0 + \alpha\, \eta(x,y)$ $\quad$ α = small parameter

η = arbitrary function which $= 0$ on ∂R

$$\varphi = \varphi\left(f_0 + \alpha\eta,\ f_{0x} + \alpha\eta_x,\ f_{0y} + \alpha\eta_y,\ x, y\right)$$

$$= \varphi(f_0) + \alpha\eta \left.\frac{\partial\varphi}{\partial f}\right|_0 + \alpha\eta_x \left.\frac{\partial\varphi}{\partial f_x}\right|_0 + \alpha\eta_y \left.\frac{\partial\varphi}{\partial f_y}\right|_0$$

$$I[f] = I[f_0] + \alpha \int_R dx\,dy\, \eta(x,y) \underbrace{\left[\frac{\partial\varphi}{\partial f} - \frac{\partial}{\partial x}\frac{\partial\varphi}{\partial f_x} - \frac{\partial}{\partial y}\frac{\partial\varphi}{\partial f_y}\right]}_{=0 \text{ for arbitrary } \eta(x,y)}$$

$$\rightarrow \quad \frac{\partial\varphi}{\partial f} - \frac{\partial}{\partial x}\frac{\partial\varphi}{\partial f_x} - \frac{\partial}{\partial y}\frac{\partial\varphi}{\partial f_y} = 0$$

In 3d - extra term $-\dfrac{\partial}{\partial z}\dfrac{\partial\varphi}{\partial f_z}$