Lagrange Eqs hold in any "good" set of generalized coordinates

proof: suppose $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$ for $\{q_1 \dots q_n\}$

and $\{s_i\}$ = alternate set with $q_i = q_i(s_1 \dots s_n, t)$

notice $\dot{q}_j = \sum_k \frac{\partial q_j}{\partial s_k}\dot{s}_k + \frac{\partial q_j}{\partial t} \Rightarrow \frac{\partial \dot{q}_j}{\partial \dot{s}_i} = \frac{\partial q_j}{\partial s_i}$

and $\frac{d}{dt}\frac{\partial q_j}{\partial s_i} = \sum_k \frac{\partial^2 q_j}{\partial s_i \partial s_k}\dot{s}_k + \frac{\partial^2 q_j}{\partial s_i \partial t} = \frac{\partial \dot{q}_j}{\partial s_i}$

same as in original derivation

Then $\frac{\partial L}{\partial \dot{s}_i} = \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial \dot{s}_i} = \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{\partial q_k}{\partial s_i}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_i}\right) = \sum_k \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\frac{\partial q_k}{\partial s_i} + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\frac{\partial q_k}{\partial s_i}\right]$$

$$= \sum_k \left[\frac{\partial L}{\partial q_k}\frac{\partial q_k}{\partial s_i} + \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial s_i}\right]$$

$$= \frac{\partial L}{\partial s_i}$$

Other general result: if $L \to L + f(t)$

no change in eqs. of motion

$\to L$ highly non-unique

Cyclic coordinates:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} = \frac{\partial L}{\partial q_i} = \begin{cases} \partial V/\partial q_i \\ Q_i \end{cases} = \text{generalized forces}$$

natural to call this a "generalized momentum"

$$p_i \equiv \frac{\partial L}{\partial \dot q_i} \quad \to \quad \dot p_i = \frac{\partial L}{\partial q_i} = \text{force}$$

e.g. $L = \frac{1}{2}m\dot q^2 - V(q)$ $\quad p = m\dot q$ = usual momentum

$L = \frac{1}{2}I\dot\theta^2 - V(\theta)$ $\quad p = I\dot\theta$ = angular momentum

$L = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) - V(r,\theta)$ $\quad p_r = m\dot r$ linear momentum

$p_\theta = mr^2\dot\theta$ angular "

$L = \frac{1}{2}m\underline{v}^2 - q\varphi + q\underline{v}\cdot\underline{A}$ $\quad \underline{p} = m\underline{v} + q\underline{A}$!

This is why the original $\underline{F} = m\underline{a}$ derivation fails for magnetic fields

If q_i does not appear in L then p_i = constant

q_i = "cyclic coordinate"

e.g. $V(r,\theta) \to V(\theta) \to p_\theta$ = constant

$L = \frac{1}{2}m(\dot x^2 + \dot y^2) - mgy \to p_x$ = constant

Translation invariance:

N-particle system with $V(\{\underline{r}_i\}) = \sum_{i<j} V_2(\underline{r}_i - \underline{r}_j)$

L is unchanged if all $\underline{r}_i(t) \to \underline{r}_i(t) + \underline{\Delta}$, $\Delta = \text{constant}$

$$L(\underline{r}_i, \dot{\underline{r}}_i) = L(\underline{r}_i + \underline{\Delta}, \dot{\underline{r}}_i)$$

so $\dfrac{\partial L}{\partial \underline{\Delta}} = 0 = \sum_i \dfrac{\partial L}{\partial \underline{r}_i} = \sum_i \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\underline{r}}_i} = \dfrac{d}{dt}\sum_i \underline{p}_i = \dfrac{d\underline{P}}{dt}$

$\Rightarrow$ translation invariance $\Longleftrightarrow$ conservation of momentum!

Conservation of energy:

$$\frac{dL}{dt} = \sum_i \Big[\underbrace{\frac{\partial L}{\partial q_i}}_{dp_i/dt} \dot{q}_i + \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} \ddot{q}_i \Big] + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt} \sum_i p_i \dot{q}_i + \frac{\partial L}{\partial t}$$

or $\dfrac{d}{dt}\Big(\underbrace{\sum_i p_i \dot{q}_i - L}_{\equiv H \,=\, \text{"Hamiltonian function"}} \Big) = \dfrac{\partial L}{\partial t}$

$\Rightarrow$ If L is not an explicit function of time, H = constant

What's H mean?

example: $L = \sum \frac{1}{2} m_i \underline{v}_i^2 - V(\underline{r}_i)$

$p_i = m_i \underline{v}_i$ so $H = \sum (m_i \underline{v}_i) \cdot \underline{v}_i - L = \sum \frac{1}{2} m_i \underline{v}_i^2 + V$

$= E$

$$\frac{dE}{dt} = \sum_i m_i \underline{v}_i \cdot \dot{\underline{v}}_i - \sum_i \frac{\partial L}{\partial \underline{r}_i} \cdot \dot{\underline{r}}_i = 0 \qquad E = H = \text{constant}$$

generalization: suppose $\underline{r}_i = \underline{r}_i(q_1 \dots q_n)$ with no t

$\to x_\alpha = x_\alpha(q_1 \dots q_n)$ where $\underline{x} = (r_{1x}\, r_{1y}\, r_{1z}\, r_{2x}\, r_{2y}$ etc.$)$

$$T = \frac{1}{2} \sum_i m_i \dot{\underline{r}}_i^2 = \frac{1}{2} \sum_\alpha m_\alpha \dot{x}_\alpha^2 = \frac{1}{2} \sum_\alpha m_\alpha \left(\sum_i \frac{\partial x_\alpha}{\partial q_i} \dot{q}_i \right)^2$$

$$= \frac{1}{2} \sum_{ij} \left(\sum_\alpha m_\alpha \frac{\partial x_\alpha}{\partial q_i} \frac{\partial x_\alpha}{\partial q_j} \right) \dot{q}_i \dot{q}_j$$

$$= \frac{1}{2} \sum_{ij} M_{ij}(\underline{q}) \dot{q}_i \dot{q}_j \qquad M_{ij} = \text{"mass matrix"}$$

If V depends only on q_i then $p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_j M_{ij} \dot{q}_j$

and $H = \sum_i p_i \dot{q}_i - L = \sum \frac{1}{2} M_{ij} \dot{q}_i \dot{q}_j + V = T + V = E$!

Is this constant?

$$\frac{dH}{dt} = \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) - \sum_i \left(\underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i} \dot{q}_i + \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} \ddot{q}_i \right) - \frac{\partial L}{\partial t}$$

$$= -\frac{\partial L}{\partial t}$$

So $\quad L = T - V \quad H = \sum p_i \dot{q}_i - L$

If V is independent of t and $q_i = q_i(\underline{r})$ only (no t)

then $H = E =$ constant

If only $q_i = q_i(\underline{r})$ then $H = E$ and

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \neq 0$$

If only L independent of t then

$H \neq E$ in general but $\frac{dH}{dt} = 0$, $\frac{dE}{dt} = ?$

In the 1st case: no explicit time dependence anywhere

= "time translation invariance"

$\Longleftrightarrow$ conservation of energy

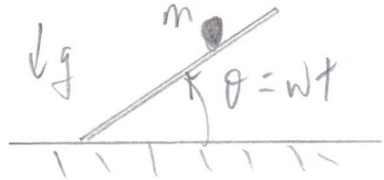
Will show later:

rotational invariance $\Longleftrightarrow$ cons. of angular momentum

General Rule:

invariance = symmetry $\Longleftrightarrow$ conservation law

7.33 Mass on plate pivoting away from horizontal at ω



use polar coord

$\underset{\sim}{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$\rightarrow \dot{r}\hat{r} + r\omega\hat{\theta}$

$T = \frac{1}{2}m\underset{\sim}{v}^2 = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$, $V = mgh = mgr\sin\omega t$

$L = T - V$, $p = m\dot{r}$

$\frac{d}{dt}(m\dot{r}) = mr\omega^2 - mg\sin\omega t$

or $\ddot{r} - \omega^2 r = -g\sin\omega t$

motion: Take $r(0) = r_0$, $\dot{r}(0) = 0$ start at rest at r_0

$r(t) = r_p(t) + r_h(t) = r_p(t) + Ae^{\omega t} + Be^{-\omega t}$

as with forced harmonic oscillator, try $r_p = C\sin\omega t$

$\rightarrow -\omega^2 C\sin\omega t - \omega^2 C\sin\omega t = -g\sin\omega t$

$\rightarrow C = g/2\omega^2$

$r(t) = \frac{g}{2\omega^2}\sin\omega t + Ae^{\omega t} + Be^{-\omega t}$

$r_0 = A + B$, $0 = \frac{g}{2\omega} + \omega(A - B)$

$\rightarrow A = \frac{r_0}{2} - \frac{g}{4\omega^2}$ $B = \frac{r_0}{2} + \frac{g}{4\omega^2}$

or $r(t) = r_0\cosh\omega t + \frac{g}{2\omega^2}(\sin\omega t - \sinh\omega t)$

$\xrightarrow[t\to\infty]{} \left(\frac{r_0}{2} - \frac{g}{4\omega^2}\right)e^{\omega t}$

This is a straightforward calculation:

geometry $\rightarrow$ generalized coordinates $\xrightarrow[\text{add } V]{}$ L

$\rightarrow$ eq of motion $\rightarrow$ trajectory

Physics: the rotation tries to push the mass outwards

fixed r would require centripetal force

gravity tries to pull the mass down to $r=0$

If g is too small $\left(g/4\omega^2 < r_0/2 \right)$ $\ddot{r} > 0$ - moves out

g is too big $\left(g/4\omega^2 > r_0/2 \right)$ $\ddot{r} < 0$ - moves in

Conservation of E, H ?

$$\frac{\partial L}{\partial t} \neq 0 \quad \text{so} \quad H = (m\dot{r})\dot{r} - L$$

$$= \frac{1}{2}m\dot{r}^2 - \frac{1}{2}m\omega^2 r^2 + mgr\sin\omega t$$

$$\neq \text{const}$$

$H \neq E$

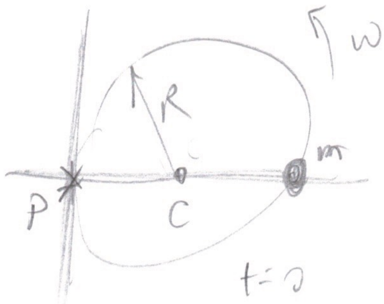
$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}m\left(\dot{r}^2 + \omega^2 r^2\right) + mgr\sin\omega t\right]$$

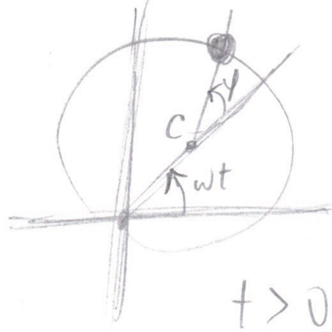
$$= m\dot{r}\underbrace{\left(\ddot{r} + \omega^2 r + g\sin\omega t\right)}_{=0} + mgr\omega\cos\omega t$$

$$\neq 0$$

Both L and the transform to generalized coord depend on t.

2.35 Bead slides w/o friction on a circular loop rotating about a point on its circumference





position of bead is

$$\underline{r} = \underline{r}_c + \underline{r}' = \Big(R\cos\omega t + R\cos(\omega t+\varphi),\ R\sin\omega t + R\sin(\omega t+\varphi)\Big)$$

$$\dot{\underline{r}} = R\Big(-\omega\sin\omega t - (\omega+\dot\varphi)\sin(\omega t+\varphi),\ \omega\cos\omega t + (\omega+\dot\varphi)\cos(\omega t+\varphi)\Big)$$

$$\begin{cases} T = \tfrac{1}{2}m\dot{\underline{r}}^2 = \tfrac{1}{2}mR^2\left[\omega^2 + (\omega+\dot\varphi)^2 + 2\omega(\omega+\dot\varphi)\cos\varphi\right] \\ V = 0 \end{cases}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\varphi} = \frac{\partial L}{\partial\varphi} \rightarrow \frac{d}{dt}\left[\omega+\dot\varphi+\omega\cos\varphi\right] = -\omega(\omega+\dot\varphi)\sin\varphi$$

or $\ddot\varphi - \cancel{\omega\dot\varphi\sin\varphi} = -\omega^2\sin\varphi - \cancel{\omega\dot\varphi\sin\varphi}$

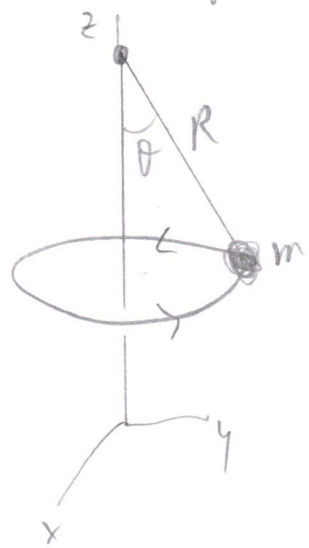
$\rightarrow$ simple pendulum

$$H = p\dot\varphi - L = mR^2\left(\tfrac{1}{2}\dot\varphi^2 - \omega^2\cos\varphi\right) \quad \checkmark \ \frac{\partial L}{\partial t} = 0$$

$$\frac{dH}{dt} = mR^2\dot\varphi\left(\ddot\varphi + \omega^2\sin\varphi\right) = 0$$

$H \neq E$ because $\underline{r}(\varphi)$ involves t

7.40 Spherical pendulum



use spherical coordinates

$$\underline{v} = R\dot{\theta}\,\hat{\theta} + R\sin\theta\,\dot{\varphi}\,\hat{\varphi}$$

$$T = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)$$

$$V = mgR(1-\cos\theta)$$

$$L = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) + mgR\cos\theta + \text{const.}$$

$$\theta: \quad \frac{d}{dt}\left(mR^2\dot{\theta}\right) = mR^2 \sin\theta\cos\theta\,\dot{\varphi}^2 - mg\sin\theta$$

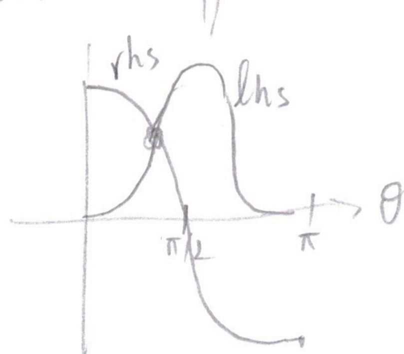
$$\varphi: \quad \frac{d}{dt}\Big(\underbrace{mR^2\sin^2\theta\,\dot{\varphi}}_{=L_z}\Big) = 0 \quad \text{or} \quad \frac{d}{dt}\left(L_z\right) = 0$$

usual pendulum when φ = constant

eliminate $\dot{\varphi}$ in favor of l_z: $\quad \dot{\varphi} = l_z / mR^2\sin^2\theta$

$$\rightarrow \quad R\ddot{\theta} = -g\sin\theta + \frac{L_z^2}{m^2R^3}\frac{\cos\theta}{\sin^3\theta}$$

Possible equilibria when $\ddot{\theta} = 0$, $\quad g\sin^4\theta = \frac{L_z^2}{m^2R^3}\cos\theta$



one solution in $0 < \theta_0 < \frac{\pi}{2}$

For stability write $\theta = \theta_0 + \delta\theta$

+ expand to $O(\delta\theta)$

$$g\sin^4\theta = g\left(\sin\theta_0 + \delta\theta\cos\theta_0 + \cdots\right)^4$$

$$= g\sin^4\theta_0 + 4g\sin^3\theta_0\cos\theta_0\,\delta\theta$$

$$\frac{l_z^2}{m^2R^3}\cos\theta = \frac{l_z^2}{m^2R^3}\left(\cos\theta_0 - \sin\theta_0\,\delta\theta\right)$$

$$R\ddot{\theta} \to R\,\delta\ddot{\theta} = -4g\sin^3\theta_0\cos\theta_0\,\delta\theta - \frac{l_z^2}{m^2R^3}\sin\theta_0\,\delta\theta$$

$$= (\text{negative number})\cdot\delta\theta$$

so θ_0 = stable equilibrium

Motion: $\dot{\varphi}$ = constant, $\theta = \theta_0$ + small osc about it

→ m moves in a horizontal plane at θ_0 with small osc. in θ + $\dot{\varphi}$

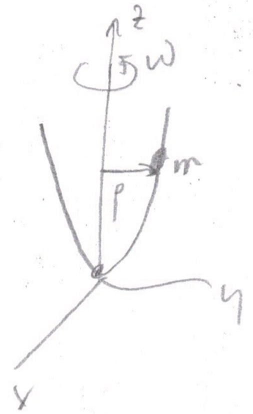
$$H = p_\theta\dot{\theta} + p_\varphi\dot{\varphi} - L$$

$$= (mR^2\dot{\theta})\dot{\theta} + (mR^2\sin^2\theta\,\dot{\varphi})\dot{\varphi} - L$$

$$= T + V = E$$

$\frac{dH}{dt} = 0$ because $L \neq L(t)$

7.41 Bead on a rotating parabolic wire



wire $z = k\rho^2$

rotates about $\hat{z}$ at ω

no friction

Use cylindrical coordinates

$$L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) - mgz$$

$$\rightarrow \frac{1}{2}m\left[\left(1+4k^2\rho^2\right)\dot{\rho}^2 + \omega^2\rho^2\right] - mgk\rho^2$$

after using $z = k\rho^2$ and $\varphi = \omega t$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\rho}} = \frac{\partial L}{\partial\rho} \rightarrow \frac{d}{dt}\left[\left(1+4k^2\rho^2\right)\dot{\rho}\right] = 4k^2\dot{\rho}^2\rho + \omega^2\rho - 2gk\rho$$

or $\left(1+4k^2\rho^2\right)\ddot{\rho} + 4k^2\rho\dot{\rho}^2 = \left(\omega^2 - 2gk\right)\rho$

Possible equilibria when rhs $= 0$

$\rightarrow \rho = 0$ or $\omega^2 = 2gk$

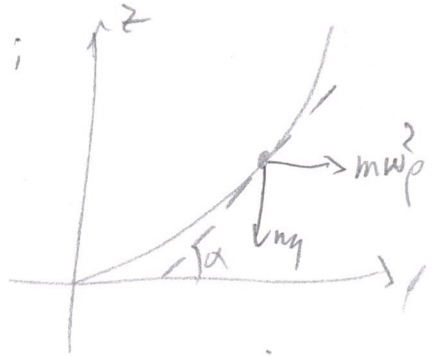
(1) (2)

for (1) test stability by looking at $\rho \cong 0$, neglect $O(\rho^2)$

$$\ddot{\rho} = \left(\omega^2 - 2gk\right)\rho$$

stable for $\omega^2 < 2gk$

significance :



$2k\rho = z'(\rho) = \tan\alpha$

$\rho\omega^2 = 2gk\rho$

$\rightarrow \rho\omega^2 \cos\alpha = g \tan\alpha$

or centrifugal = gravity tangent to wire

(normal force compensates in $\perp$ direction)

So $\omega^2 < 2gk \rightarrow$ gravity beats centrifugal

(2) $\ddot{\rho} = -\dfrac{4k^2\dot{\rho}^2}{1+4k^2\rho^2}\rho$

If m is displaced but not moving $\rightarrow$ no motion

Suppose $\dot{\rho} \neq 0$ but small : $\ddot{\rho} = -(\,)\rho$: stable

If bead starts at the bottom + $\omega^2 > 2gk$ it will be unstable start to move up, subsequent motion needs further analysis.