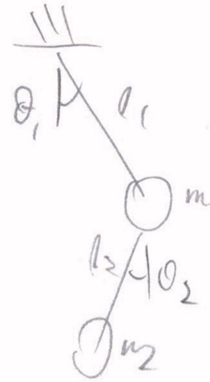
Disadvantages of $\underline{F} = m\underline{a}$: always true, but ...

1. Awkward coordinates :

"double pendulum"

e.g.



Normal treatment $m\ddot{x}_2 = T_{12}\cos\theta_2$

$$m_2\ddot{y}_2 = -m_2 g + T_{12}\cos\theta_2$$

$$\cos\theta_2 = \frac{x_1 - x_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}$$

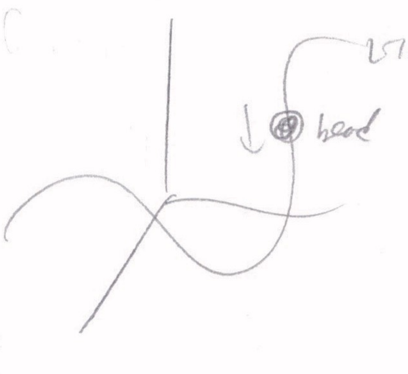
etc.

Would prefer to work directly in terms of θ_1 & θ_2, not $\underline{r}_i$, but $\underline{a}$ is messy in polar coord (very messy in 3d)

2. Constraint forces:

T_{12} = tension force may not be interesting at all

Similar situation - bead on a wire



If wire is $x = x(z)$, $y = y(z)$ then get

$$m\ddot{z} = -mg + N_z$$
$$m\ddot{y} = N_y$$
$$m\ddot{x} = N_x$$

constraint forces

$\underline{N}$ = force exerted on bead by wire

= force constraining bead to move along $y = y(z)$, $x = x(z)$

: needed to solve problem, but not that interesting

3. Translation vs. rotation

look at object rotating about the origin in 2d, under $V(x,y)$

$$\underline{\Gamma}_0 = \underline{r}\times\underline{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & 0 \\ -\frac{\partial V}{\partial x} & -\frac{\partial V}{\partial y} & 0 \end{vmatrix} = \hat{z}\left(-x\frac{\partial V}{\partial y} + y\frac{\partial V}{\partial x}\right)$$

But $V(x,y) \leftrightarrow V(r,\theta)$

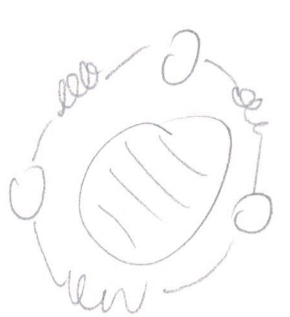
$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \theta} \xrightarrow[\substack{x = r\cos\theta \\ y = r\sin\theta}]{} \frac{\partial V}{\partial x}\left(-r\sin\theta\right) + \frac{\partial V}{\partial y}\left(r\cos\theta\right)$$

$$= -\left(x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x}\right)$$

so $\Gamma_z = -\frac{\partial V}{\partial \theta} = I\ddot{\theta}$ and $F_x = -\frac{\partial V}{\partial x} = m\ddot{x}$

: treat (x,θ), (m,I), (F,N) on the same footing

4. Obvious symmetries:



3 masses on outside of a cylinder connected by springs

obvious variables are $\theta_1, \theta_2, \theta_3$

" symmetry $\theta_i \to \theta_i + \Delta\theta$

$\to$ better variables are $\theta_1 - \theta_2$, $\theta_2 - \theta_3$

but need individual eqs first

5. Continuum systems

what is F on a blob of stuff?

Strategy: (1) Show $\underline{F} = m\underline{a} \Leftrightarrow$ Lagrange's Eqs.

($1\frac{1}{2}$) Examples $\rightsquigarrow$

(2) Show LE follows from a "Variational Principle"

- min/max of a certain function

(3) Adopt V.P. as the starting point for fields.

(In the book, the math for #2 - Calculus of Variations - is first, then the equivalence argument goes back & forth, #3 illustrated at the end.)

Example: 3d particle in conservative force field $m\ddot{\underline{r}} = -\frac{\partial V}{\partial \underline{r}}$

define Lagrangian $L = T - V = \frac{1}{2}m\dot{\underline{r}}^2 - V(\underline{r})$

then $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{r}}}\right) = \frac{\partial L}{\partial \underline{r}} \rightarrow \frac{d}{dt}(m\dot{\underline{r}}) = -\frac{\partial V}{\partial \underline{r}}$

: trivial here, but L can be expressed in terms of any reasonable coordinates, some q_i, and any constraints are handled by elimination at start & constraint forces do not appear.

LE Start from $m_i \ddot{\underline{r}}_i = \underline{f}_i = -\frac{\partial V}{\partial r_i}$ + constraint forces

+ relabel variables $\{\underline{r}_1 \cdots \underline{r}_N\} \to \{x_1 x_2 \cdots x_{3N}\}$

+ Newton $\to$ $m_\alpha \ddot{x}_\alpha = F_\alpha$ $\quad \alpha = 1 \cdots 3N$

$m_\alpha = \{m_1 m_1 m_1 m_2 m_2 m_2 \cdots\}$

The KE is $T = \sum_1^N \frac{1}{2} m_i \dot{\underline{r}}_i^2 = \sum_1^{3N} \frac{1}{2} m_\alpha \dot{x}_\alpha^2$ +

Newton: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_\alpha}\right) = F_\alpha$

$n \le 3N$

Go to the 'generalized coordinates' $\{q_1 \cdots q_n\}$, which are unconstrained + completely specify the configuration. These could be lengths or angles. In doing so eliminate redundant variables by hand; if not possible (transcendental eq.) use Lagrange multiplier method below.

So new coords are $q_i = q_i(x_1 \cdots x_{3N}, t)$ $\quad i = 1, 2, \cdots, n \le 3N$

+ these eqs should be invertible $x_\alpha = x_\alpha(q_1 \cdots q_n, t)$

Then $\dot{x}_\alpha = \sum_{j=1}^{n} \frac{\partial x_\alpha}{\partial q_j} \dot{q}_j + \frac{\partial x_\alpha}{\partial t}$ by the chain rule

So $\frac{\partial \dot{x}_\alpha}{\partial \dot{q}_i} = \frac{\partial x_\alpha}{\partial q_i}$ + $\frac{d}{dt}\frac{\partial x_\alpha}{\partial q_i} = \frac{\partial \dot{x}_\alpha}{\partial q_i}$

1st is obvious, 2nd needs proof:

$$\text{lhs of } 2 = \frac{d}{dt}\frac{\partial x_\alpha}{\partial q_i} = \frac{\partial^2 x_\alpha}{\partial q_i \partial t} + \sum_j \frac{\partial^2 x_\alpha}{\partial q_i \partial q_j}\dot{q}_j$$

$$\text{rhs of } 2 = \frac{\partial}{\partial q_i}\left(\sum_j \frac{\partial x_\alpha}{\partial q_j}\dot{q}_j + \frac{\partial x_\alpha}{\partial t}\right) = \sum_j \frac{\partial^2 x_\alpha}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 x_\alpha}{\partial q_i \partial t}$$

---

e.g. $x = r\cos\theta \qquad y = r\sin\theta$

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\Rightarrow \frac{\partial \dot{x}}{\partial \dot{r}} = \cos\theta = \frac{\partial x}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial x}{\partial r}\right) = \frac{d}{dt}(\cos\theta) = -\dot{\theta}\sin\theta = \frac{\partial \dot{x}}{\partial r} \text{ ; sim. for } y$$

---

upshot is $T(\dot{x}_\alpha) \to T(q_\alpha, \dot{q}_\alpha, t)$

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_\alpha \frac{\partial T}{\partial \dot{x}_\alpha}\frac{\partial \dot{x}_\alpha}{\partial \dot{q}_i} = \sum_\alpha \frac{\partial T}{\partial \dot{x}_\alpha}\frac{\partial x_\alpha}{\partial q_i}$$

$$\& \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_\alpha \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_\alpha}\right)\frac{\partial x_\alpha}{\partial q_i} + \frac{\partial T}{\partial \dot{x}_\alpha}\underbrace{\frac{d}{dt}\left(\frac{\partial x_\alpha}{\partial q_i}\right)}_{\frac{\partial \dot{x}_\alpha}{\partial q_i}}\right]$$

$$= \sum_\alpha F_\alpha \frac{\partial x_\alpha}{\partial q_i} + \frac{\partial T}{\partial q_i}$$

$$\text{so} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \sum_\alpha F_\alpha \frac{\partial x_\alpha}{\partial q_i} = Q_i$$

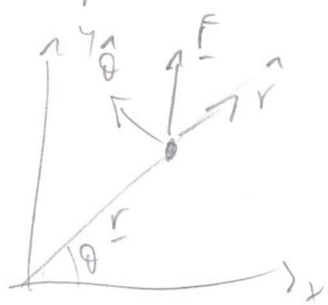
Notice $Q_i \delta q_i = \sum_\alpha f_\alpha \frac{\partial x_\alpha}{\partial q_i} \delta q_i$

$= \delta x_\alpha$: change in x_α resulting from a change in q_i at fixed t.

= work done when $q_i \to q_i + \delta q_i$

so Q_i = generalized force in direction q_i

Back to polar coord example:



$$Q_r = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r}$$

$$= f_x \cos\theta + f_y \sin\theta = \underline{F} \cdot \hat{r}$$

= force along $\hat{r}$

$$Q_\theta = f_x \frac{\partial x}{\partial \theta} + f_y \frac{\partial y}{\partial \theta} = - f_x r \sin\theta + f_y r \cos\theta$$

$$= \left| \begin{matrix} \hat{x} & \hat{y} & \hat{z} \\ r\cos\theta & r\sin\theta & 0 \\ F_x & F_y & 0 \end{matrix} \right| = |\underline{r} \times \underline{F}| = \theta\text{-torque}$$

: gen'l coord is length or angle so gen'l force is force or torque.

If the force is conservative $\underline{F}_i = -\frac{\partial V}{\partial \underline{r}_i}$ or $f_\alpha = -\frac{\partial V}{\partial x_\alpha}$

then $Q_i = \sum_\alpha f_\alpha \frac{\partial x_\alpha}{\partial q_i} = -\sum \frac{\partial V}{\partial x_\alpha} \frac{\partial x_\alpha}{\partial q_i} = -\frac{\partial V}{\partial q_i}$

If V is velocity-independent $\frac{\partial V}{\partial \dot{q}_i} = 0 = \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_i}$

so $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i = -\frac{\partial V}{\partial q_i} \Rightarrow$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \quad i = 1, 2 \ldots n, \quad \text{where } L \equiv T - V}$$

If V does depend on velocity, either

(1) leave it as Q_i on rhs (e.g. friction)

(2) sometimes you still get L.E.

e.g. E+M $\quad L = \frac{1}{2} m \underline{v}^2 - q\varphi + q\underline{A}\cdot\underline{v}$

where $\underline{E} = -\nabla\varphi - \frac{\partial \underline{A}}{\partial t}$ + $\underline{B} = \nabla \times \underline{A}$

(see section 7.9 in Taylor)

Constraint forces – may not be conservative, and a main motivation for this was to avoid them

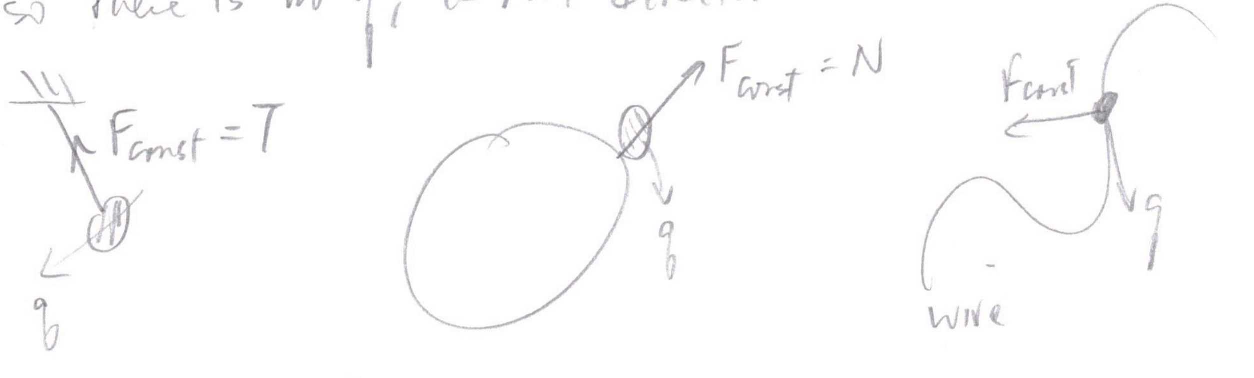
→ They do not contribute: Q_i = force in direction q_i

but by definition the q_i are not constrained.

A constraint force prevents motion in its direction

so there is no q_i in that direction

e.g.



Related idea – "principle of virtual work"

virtual displacement $\delta \underline{r}_i$ = instantaneous motion consistent with constraints

$$0 = \sum_i \left(\underline{F}_i - \dot{\underline{p}}_i \right) \cdot \delta \underline{r}_i = \sum_i \left(\underline{F}_i^{appl} - \dot{\underline{p}}_i \right) \cdot \delta \underline{r}_i + \underbrace{\sum \underline{F}_i^{const} \cdot \delta \underline{r}_i}_{0}$$

sometimes used to derive L.E.

— x —

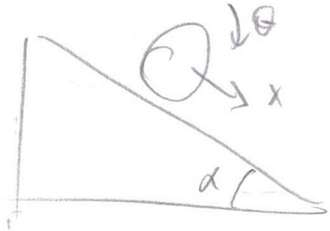
Note: (1) Only as many Lagrange Eqs as degrees of freedom

(2) Only q_i + $\dot{q}_i$ are needed for setup – not $\ddot{q}_i$
Also $T + V$ easier to find than $\underline{F}$ (often)

(3) Precise form of q_i not needed in derivation ⇒
LE apply for any set of generalized coordinates
(explicit proof later)

Examples:

1. Ball rolling w/o slip on an incline



$$T = T_{cm} + T'$$
$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$$

rolling constraint: $\dot{x} = R\dot{\theta}$

$$\rightarrow \quad T = \frac{1}{2}\left(m + I/R^2\right)\dot{x}^2$$

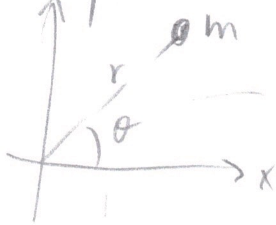
$$V = V_{cm} = -mgx\sin\alpha$$

$$L = \frac{1}{2}\left(m + I/R^2\right)\dot{x}^2 + mgx\sin\alpha$$

$$\rightarrow \quad \frac{d}{dt}\left[\left(m + I/R^2\right)\dot{x}\right] = mg\sin\alpha$$

or $\left(m + I/R^2\right)\ddot{x} = mg\sin\alpha$

2. 2d central force



$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

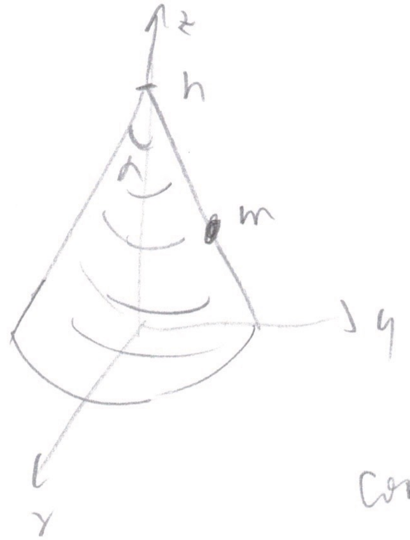
$$V = V(r)$$

$$L = T - V$$

r eq: $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \rightarrow \frac{d}{dt}\left(m\dot{r}\right) = -\frac{\partial V}{\partial r}$ — force eq

θ eq: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \rightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$ — conservation of angular momentum

3. Point mass moving on the surface of a cone



α = vertex angle

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

$\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2$ in cyl. coord.

constraint: $\tan\alpha = \dfrac{r}{h-z}$

or $z = h - r/\tan\alpha$

→ eliminate constrained variable z

$$L = \frac{m}{2}\Big[\underbrace{\left(1 + 1/\tan^2\alpha\right)}_{= 1/\sin^2\alpha}\dot{r}^2 + r^2\dot{\theta}^2\Big] + \frac{mgr}{\tan\alpha} + \text{const}$$

r eq: $\dfrac{m}{\sin^2\alpha}\ddot{r} = \dfrac{mg}{\tan\alpha} + mr\dot{\theta}^2$

θ eq: $\dfrac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$ no torque so const ang. mom.

check for θ fixed

$m\ddot{x} = mg\cos\alpha$

$r = x\sin\alpha$

$\rightarrow m\ddot{r} = mg\sin\alpha\cos\alpha$ ✓

θ not fixed → orbits (later chapter)

Bead on a wire

wire : $x = x(z) \quad y = y(z)$

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

$$\dot{x} = x'(z)\dot{z} \quad \dot{y} = y'(z)\dot{z}$$

$$\rightarrow L = \frac{m}{2}\underbrace{\left[1 + x'(z)^2 + y'(z)^2\right]}_{f(z)}\dot{z}^2 - mgz$$

$$\frac{d}{dt}\left(\cancel{m} f(z)\dot{z}\right) = \frac{\partial}{\partial z}\left(\frac{\cancel{m}}{2} f(z)\dot{z}^2\right) - mg$$

$$f(z)\ddot{z} + f'(z)\dot{z}^2 = \frac{1}{2}f'(z)\dot{z}^2 - mg$$

$$\rightarrow f(z)\ddot{z} + \frac{1}{2}f'(z)\dot{z}^2 + mg = 0$$

Probably hard to solve but just 1 ODE in 1 unknown

If the wire is straight, $x = \alpha z \quad y = \beta z$, then

$$f = 1 + \alpha^2 + \beta^2 = \text{const}$$

$$\ddot{z} = -mg/f > -mg \quad : \text{slower fall.}$$

Clarification about forces

we had $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} = \sum_\alpha F_\alpha \frac{\partial x_\alpha}{\partial q_i} = Q_i$

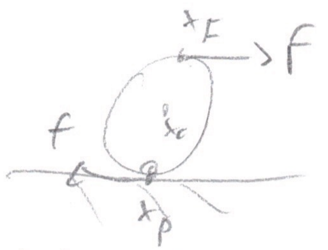
if $F_\alpha = -\frac{\partial V}{\partial x_\alpha}$ then $Q_i = -\sum \frac{\partial V}{\partial x_\alpha}\frac{\partial x_\alpha}{\partial q_i} = -\frac{\partial V}{\partial q_i}$

and $L = T - V$ = previous case

if not : notice $\frac{\partial x_\alpha}{\partial q_i}$ ~ direction cosine

so start from F_α

e.g. in the



problem

the work done when $x_c \to x_c + \delta x_c$ is $F \cdot \delta x_F = 2F\,\delta x_c$

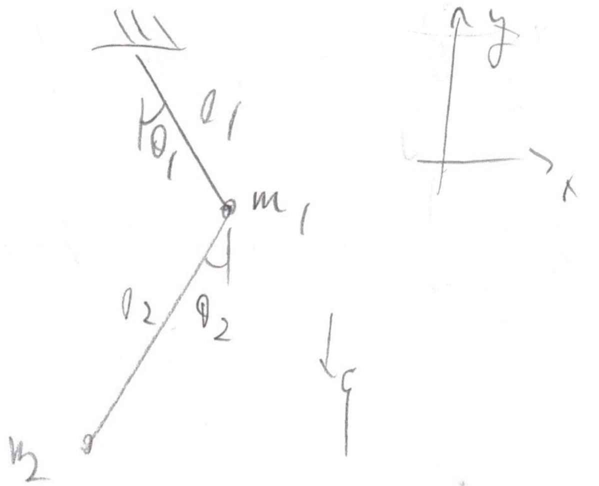
which should $= Q \cdot \delta x_c$ $\therefore Q = 2F$

so $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot x}\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot x}\left[\frac{1}{2}m\dot x^2 + \frac{1}{2}I(\dot x/R)^2\right]\right)$

$= \frac{3}{2}m\ddot x = 2F$

which is the same thing.

4. Double pendulum



$x_1 = l_1 \sin\theta_1$

$y_1 = l_1(1-\cos\theta_1)$

$T_1 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$

$V_1 = +m g y_1$

$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$

$y_2 = l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)$

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} m_2 \left[(l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)^2 + (l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2)^2 \right] \\ &= \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] \end{aligned}$$

$V_2 = +m_2 g y_2$

$$\frac{d}{dt}\left(m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + 2 m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

$$= - m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1 - m_2 g l_2 \sin\theta_2$$

$$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

+ similar eq for #2

$\Downarrow$

Hopeless ness as is.

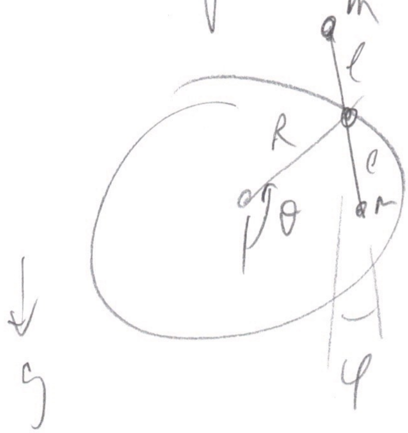
Small ang limit $|\theta_i| \ll 1$ $\sin\theta \to \theta$ $\cos\theta \to 1$

$$\frac{d}{dt}\left[(m_1+m_2)\,\dot{\theta}_1 + 2m_2 l_1 l_2 \dot{\theta}_2 \right] = -(m_1+m_2) g l_1 \theta_1 - m_2 g l_2 \theta_2$$

+ θ_2 - eqn.

: coupled linear eqs → Chapter 11.

5. Bar of length $2l$ with mass m at each end, center rotates freely about point on rim of massless disk which itself rotates freely in gravity



θ = angle of pivoting point wrt vertical

φ = angle of rod "

$$T = T_{bar}^{\;+T_{disc}} = T_{cm} + T' + T_{disc}$$

$$= \tfrac{1}{2}(2m)(R\dot\theta)^2 + 2\cdot\tfrac{1}{2} m l^2 \dot\varphi^2 + \tfrac{1}{2} I \dot\theta^2$$

$$V = 2mg \cdot y_{cm} = 2mg \cdot R(1-\cos\theta)$$

θ: $\frac{d}{dt}\left((2mR^2+I)\dot\theta\right) = -2mgR\sin\theta$

or $\ddot\theta \propto -\sin\theta$: ordinary pendulum

φ: $\frac{d}{dt}\left(2ml^2\dot\varphi\right) = 0 \rightarrow \dot\varphi = \text{const}$

: free rotation (no Γ on bar!)

total ang momentum

bar + masses

about CM