First Examination - Oct. 15, 2018 – 100 points total

1. A particle of mass m moves in one dimension subject to the force $F = -a\sin(x/b)$, where a, b are positive constants.

   (a) Find the potential energy and sketch it; it is convenient to choose the integration constant so that $V = 0$ at $x = 0$. (5 points)

   (b) Identify any points of stable or unstable equilibrium, and discuss the qualitative nature of the particle motion for different values of the total energy E. (15 points)

   (c) At time $t = 0$ the particle is at rest at $x = -b$. How long does it take to return to this point? Leave your answer in the form of a definite integral. (15 points)

2. A particle of mass m and electric charge q moves in the x-y plane, attached to the origin by a spring of stiffness k, and a constant magnetic field B is applied in the z-direction.

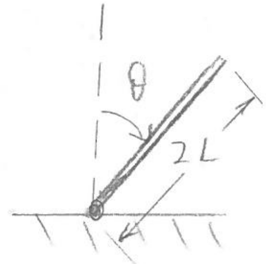
   (a) Write the equation of motion for the particle and find the possible frequencies of oscillatory motion. (20 points)

   (b) Write the general solution of the equation of motion and discuss how the constants can be found from the initial position and velocity. Do **not** do the algebra explicitly - just show that there are exactly enough independent constants to match the initial conditions. (10 points)

3. A pencil is placed vertically on its tip on a horizontal surface, and begins to fall over, as indicated in the figure below.

   (a) Assume that the tip is held in place by static friction. Find the normal and tangential components of the force that the pencil exerts on the surface, as a function of the angle θ. If you don't remember the relevant moment of inertia, just indicate the point about which it is taken. (20 points)

   (b) Now suppose the surface is friction**less** and the tip slides freely. Derive the equation of motion for θ, but you need not solve it. (10 points)



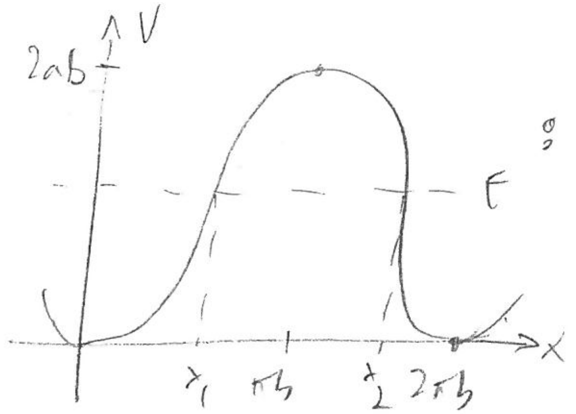
Exam 1 Solutions

1. $F = -a \sin \frac{x}{b}$ $\quad a, b > 0$

(a) Conservative? $\quad \nabla \times \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & F(x) \end{vmatrix} = 0$

$$V(x) = -\int^x dx\, F(x) = -ab \cos \frac{x}{b} + \text{constant}$$

If $V = 0$ at $x = 0$ then $V(x) = ab\left(1 - \cos \frac{x}{b}\right)$

(b) 

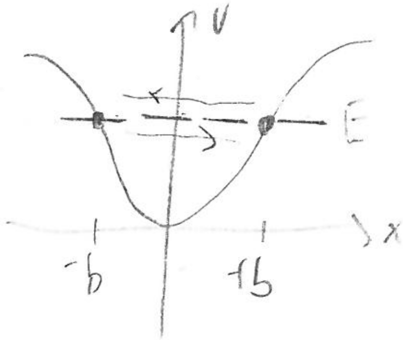
: one period of $V(x)$;
V repeats every $\Delta x = 2\pi b$

If $E > 2ab$, $\quad T = E - V > 0$ for all x,
particle can move freely $-\infty < x < +\infty$

$2ab > E > 0$: particle confined in one well
If E is as shown in the figure then
$x_1 < x < x_2$

$E < 0$: impossible

(c) If the particle is at rest at $x=-b$, then $E=V(-b)$ and



the force is to the right, so the particle can move anywhere with $V(x) < V(-b)$: up to $x=+b$, where it bounces back

The time to go from $x=-b$ to any other x is

$$t = \int_{-b}^{x} \frac{dx'}{\sqrt{\frac{2}{m}(E-V(x'))}}$$

, so the time to get back to $x=-b$ (the period) is twice the time to cross the well :

$$T = 2\int_{-b}^{b} \frac{dx'}{\sqrt{\cdots}}$$

Here $E = V(-b) = ab(1-\cos(1))$ so

$$T = 2\int_{-b}^{b} \frac{dx'}{\sqrt{\frac{2}{m}\cdot ab\left(\cos\frac{x}{b} - \cos 1\right)}}$$

$$\xrightarrow[y=\frac{x}{b}]{} \sqrt{\frac{2mb}{a}} \int_{-1}^{1} \frac{dy}{\sqrt{\cos y - \cos 1}}$$

2a) Newton's equation is $m\ddot{\underline{r}} = -k\underline{r} + q\dot{\underline{r}} \times \underline{B}$

Here $\dot{\underline{r}} \times \underline{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & 0 \\ 0 & 0 & B \end{vmatrix} = (\hat{x}\dot{y} - \hat{y}\dot{x}) B$

So $m\ddot{x} = -kx + qB\dot{y}$

$m\ddot{y} = -ky - qB\dot{x}$

For oscillatory solutions, look for $x = Ce^{i\omega t}$, $y = De^{i\omega t}$

$\rightarrow$ $\left.\begin{aligned} -m\omega^2 C &= -kC + iq\omega BD \\ -m\omega^2 D &= -kD - iq\omega BC \end{aligned}\right\}$ let $\omega_0^2 = \frac{k}{m}$, $\omega_c = \frac{qB}{m}$

$\rightarrow$ $(\omega_0^2 - \omega^2) C - i\omega\omega_c D = 0$

$(\omega_0^2 - \omega^2) D + i\omega\omega_c C = 0$

which has a solution when

$$\begin{vmatrix} \omega_0^2 - \omega^2 & -i\omega\omega_c \\ +i\omega\omega_c & \omega_0^2 - \omega^2 \end{vmatrix} = 0 = (\omega_0^2 - \omega^2)^2 - \omega^2\omega_c^2$$

or $\omega_0^2 - \omega^2 = \pm \omega_c \omega$

This has solutions $\omega = \pm \frac{\omega_c}{2} \pm \sqrt{\omega_c^2/4 + \omega_0^2}$

so the frequencies are $\omega_1 = \omega_c/2 + \sqrt{\cdots}$

$\omega_2 = \omega_c/2 - \sqrt{\cdots}$

b) The general solution for $x(t)$ can be written as

$$x(t) = \alpha \cos\omega_1 t + \beta \sin\omega_1 t + \gamma \cos\omega_2 t + \delta \sin\omega_2 t$$

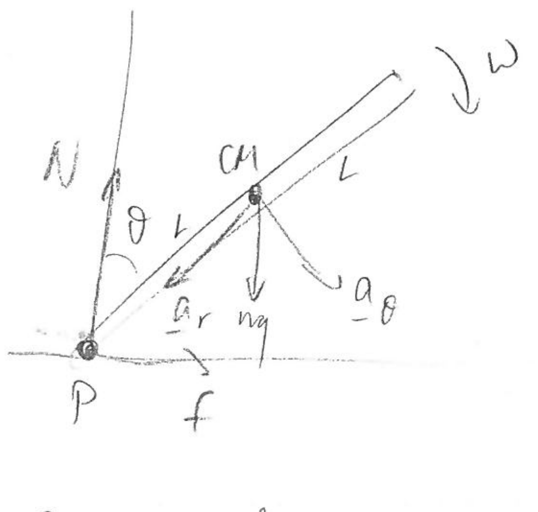
with 4 constants $\{\alpha, \beta, \gamma, \delta\}$ to be found. Note that $y(t)$ is completely determined once x is known:

$$qB\ddot{y} = \frac{d}{dt}\left(m\ddot{x} + k\dot{x}\right) = q\frac{B}{m}\left(-ky - qB\dot{x}\right)$$

$$\text{or} \quad y = -\frac{qB}{k}\dot{x} - \frac{m}{qBk}\left(m\dddot{x} + k\ddot{x}\right).$$

The 4 constants can be found by matching to the 4 initial conditions $x(0)$, $\dot{x}(0)$, $y(0)$ and $\dot{y}(0)$.

3. a)



Torque about P:

$$I_P\ddot{\theta} = mgL\sin\theta$$

$$\text{so} \quad \underline{a}_\theta = L\ddot{\theta} = \frac{mgL^2}{I_P}\sin\theta\,\hat{\theta}$$

Centripetal acceleration: $\underline{a}_r = -L\omega^2\hat{r}$

Conservation of energy: take $V=0$ when vertical

$$E = 0 = \frac{1}{2}I_P\omega^2 - mgL(1-\cos\theta)$$

so $\underset{\sim}{a_r} = -\frac{2mgL^2}{I}(1-\cos\theta)\,\hat{r}$

Force balance:

$$ma_x = m\left(a_\theta \cos\theta - a_r \sin\theta\right) = f$$

$$ma_y = m\left(-a_\theta \sin\theta - a_r \cos\theta\right) = N - mg$$

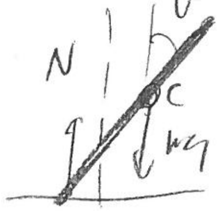
Substitute for a_r, a_θ:

$$f = \frac{m^2gL^2}{I}\sin\theta\,(3\cos\theta - 2)$$

$$N = mg - \frac{m^2gL^2}{I}(1 + 2\cos\theta - 3\cos^2\theta)$$

The tip will slip if $|f| > \mu_s N$, but this requires a numerical solution; see R. Cross, Amer. J. Phys. $\underline{74}$, 26 (2006)

b) The contact point P now accelerates, so $I_p \ddot{\theta} \neq$ torque.
Instead use the center of mass:



$$m \ddot{y}_c = N - mg$$

$$I_c \ddot{\theta} = N L \sin\theta$$

$$y_c = L \cos\theta$$

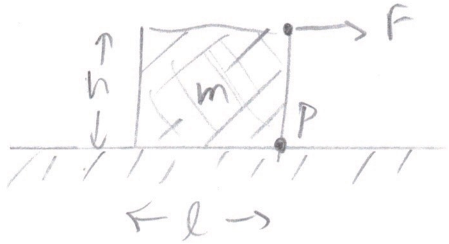
Substitute: or $$\frac{d^2}{dt^2}\left(L\cos\theta\right) = \frac{I_c \ddot{\theta}}{L \sin\theta} - mg$$

$$\vdots$$

$$\left(\sin^2\theta + \frac{I_c}{mL^2}\right)\ddot{\theta} + \sin\theta\cos\theta\,\dot{\theta}^2 = mgL\sin\theta$$

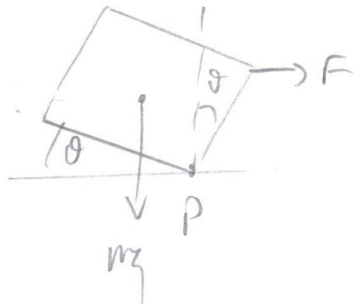
This also requires a numerical solution.

Tipping over:

1. 

$h \times l$ block on frictional surface

F needed to tip over?



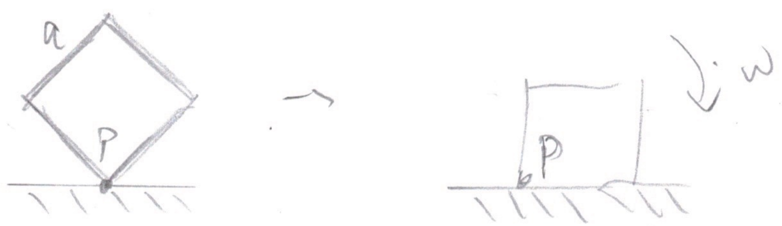
If friction holds block in place

look at torque about P

$$\Gamma_P = -F \cdot h\cos\theta + mg \cdot \frac{l}{2}\cos\theta$$

$$= \cos\theta\left(-Fh + mgl/2\right)$$

tips when $\Gamma_P < 0 \rightarrow F > mgl/2h$

2. 

before after

Cube balanced on edge tips over; ω on impact?

assume P fixed in place by friction

Before: $T = 0$, $V = mga/\sqrt{2}$

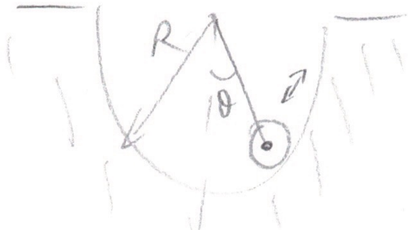
After: $T = \frac{1}{2} I_P \omega^2$, $V = mga/2$

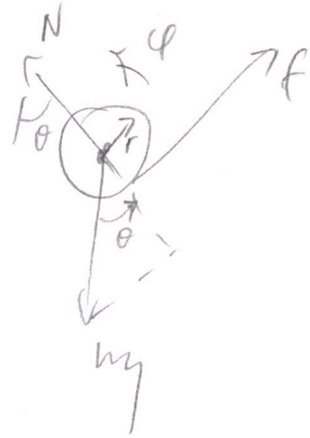
$I_c = \frac{1}{6}ma^2 + m\left(a/\sqrt{2}\right)^2$

Cons of E: $mga/\sqrt{2} = \frac{1}{2}\left(\frac{2}{3}ma^2\right)\omega^2 + mga/2$

$$\rightarrow \omega = \sqrt{\frac{3g}{2a}\left(\sqrt{2}-1\right)}$$

Ball rolls w/o slipping in a circular trough;
freq of small oscillation?





then wrt center of ball:

F_θ : $f - mg\sin\theta = ma$

Γ_z : $rf = I\ddot{\varphi}$ $\quad I = \frac{2}{5}mr^2$

kinematics: $\Delta s = R\Delta\theta = -r\Delta\varphi$, $a = (R-r)\ddot{\theta}$

so $mg\sin\theta = \frac{I}{r}\ddot{\varphi} - m(R-r)\ddot{\theta}$

or $mg\theta \simeq \frac{I}{r}\left(-\frac{R}{r}\ddot{\theta}\right) - mR\ddot{\theta}$ $\quad R \gg r$

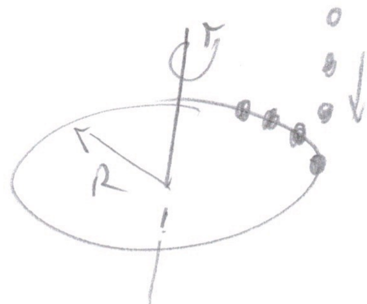
$\xrightarrow{I = \frac{2}{5}mr^2}$ $-\frac{7}{5}mR\ddot{\theta}$

: SHO with $\omega^2 = \frac{5g}{7R}$

→ slower than point particle, since of the PE goes into rotation

Sand on a turntable:

for $t<0$ turntable (m, R, I_0) rotates at ω_0 about axis



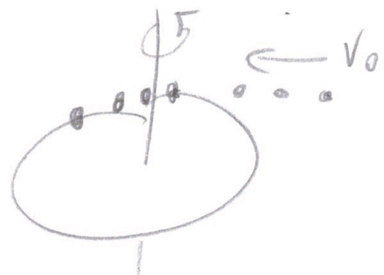
(1) Sand drops vertically onto rim

no torque: $I_0 \omega_0 = I(t)\,\omega(t)$

$$I(t) = I_0 + m(t) R^2$$

$$\rightarrow \omega(t) = \frac{\omega_0 I_0}{I_0 + m(t) R^2}$$

(2) Sand hits rim horizontally + sticks



mass $\dot{m}\Delta t$ adsorbed in time Δt

incident momentum $\dot{m}\Delta t\, v_0$

final " $(\dot{m}\Delta t)\,\omega R$

$$\Gamma = F \cdot R = \frac{\Delta p}{\Delta t} R = \frac{\dot{m}\Delta t (v_0 - \omega R)}{\Delta t} R$$

Δp of turntable

$$= \dot{m}(v_0 - \omega R) R$$

$$= \frac{d}{dt}\left(I(t)\,\omega(t)\right) = \dot{I}\omega + I\dot{\omega}$$

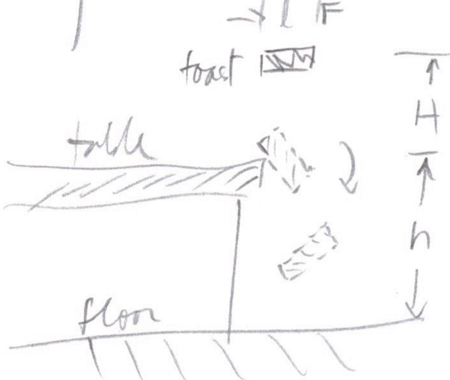
$$= \dot{m} R^2 \omega + (I_0 + m R^2)\dot{\omega}$$

$$\Rightarrow \dot{m}(v_0 - 2\omega R) R = (I_0 + m R^2)\dot{\omega}$$

$$\log(I_0 + m R^2) = -\frac{1}{2}\log(v_0 - 2\omega R) + \text{const}$$

$$\omega(t) = \frac{v_0}{2R} - \left(\frac{v_0}{2R} - \omega_0\right)\left(I_0 / I_0 + m(t) R^2\right)^2$$

Falling toast:



toast = lamina of side l

falls while horizontal

clips side of table + rotates

flips over completely at floor

Given h + l, what's H?

At table, toast moves down at $v_0 = \sqrt{2gH}$

Clip imparts momentum $\Delta p = m(v - v_0)$ to toast

$$\Gamma \Delta t = F \Delta t \cdot \frac{l}{2} = + \Delta p \cdot \frac{l}{2} = \text{ about CM}$$

$$= I_c \Delta \omega$$

or $\quad m(v - v_0)\frac{l}{2} = I_c \omega \qquad (*)$

Also $\quad \frac{1}{2} m v_0^2 = \frac{1}{2} m v^2 + \frac{1}{2} I_c \omega^2$

or $\quad m(v^2 - v_0^2) = - I_c \omega^2$

Divide: $\quad v + v_0 = -\frac{l\omega}{2}$

Rewrite $(*)$ $\quad v - v_0 = + \frac{l\omega}{6}$

$\Big\}\quad v = v_0/2, \quad \omega = -3v_0/l$

After clip: $\quad v t^* + \frac{1}{2} g t^{*2} = h \quad \to \quad t^* = \frac{1}{g}\left[-v \pm \sqrt{v^2 + 2gh}\right]$

180° rotation: $\quad \omega t^* = \pi$

$$\longrightarrow \quad \ldots \qquad H = \frac{\pi^2 l^2}{6(6h - \pi l)}$$