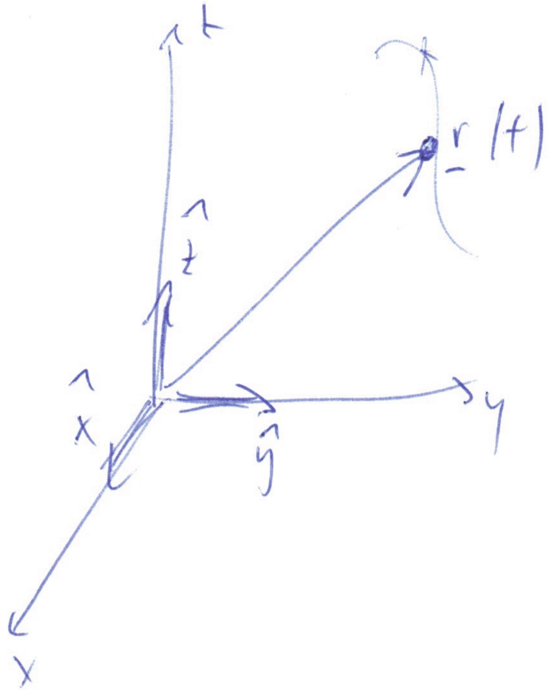
Kinematics – motion in space



$\underline{r}(t)$ = vector position of a point moving in 3d, at time t

Use fixed Cartesian coordinates

$$\underline{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$$

where $\hat{x}$ = unit vector along x-axis, $\hat{x}^2 = 1$

$\hat{y}, \hat{z}$ = " y, z axis

$x(t)$ = projection of $\underline{r}(t)$ along $\hat{x}$

$= \underline{r}(t) \cdot \hat{x}$, etc.

Notation: underline = vector, caret ($\wedge$) = unit vector

$|\underline{r}|$ = length of $\underline{r}$ $= \sqrt{x^2+y^2+z^2} = \sqrt{\underline{r}\cdot\underline{r}}$

$\hat{r}$ = "direction of $\underline{r}$" $= \underline{r}(t)/|\underline{r}(t)|$

(later: $\underline{\underline{M}}$ = double underline = matrix)

Alternate form:

$\hat{x} \to \hat{i}$ or $\hat{e}_1$; $x(t) \to r_1(t)$

$\hat{y} \to \hat{j}$ or $\hat{e}_2$; y $\quad r_2$

$\hat{z} \to \hat{k}$ or $\hat{e}_3$ $\quad z$ $\quad r_3$

so $$\underline{r}(t) = \sum_{i=1}^{3} r_i(t)\,\hat{e}_i$$

Motion:

$$\underline{v}(t) = \frac{d\underline{r}(t)}{dt} = \frac{dx(t)}{dt}\hat{x} + \frac{dy(t)}{dt}\hat{y} + \frac{dz(t)}{dt}\hat{z}$$

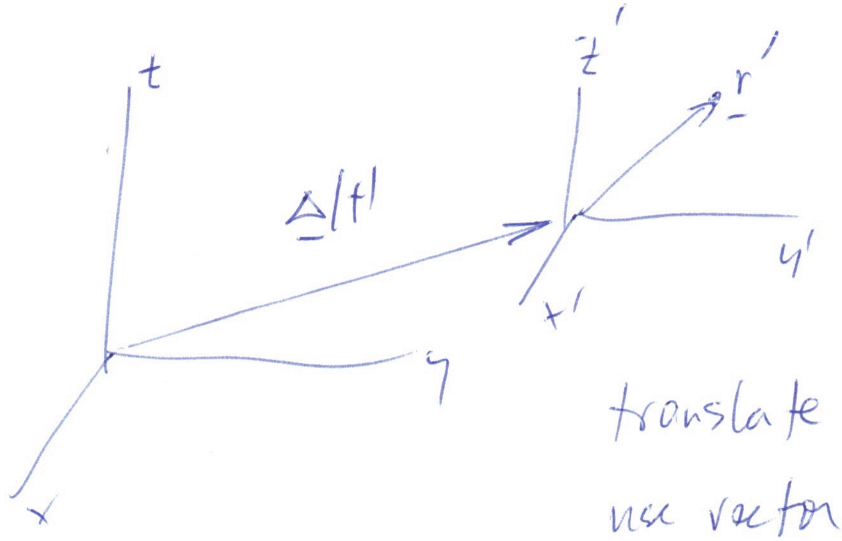
$$\dot{f}(t) \doteq \frac{df}{dt}$$

$$= \dot{\underline{r}}(t) = \sum_{i=1}^{3} \dot{r}_i(t)\,\hat{e}_i = \text{velocity}$$

$$\underline{a}(t) = \frac{d\underline{v}(t)}{dt} = \frac{d^2x(t)}{dt^2}\hat{x} + \dots = \dot{\underline{v}} = \ddot{r} = \text{acceleration}$$

(Higher derivatives wrt time not common in physics.)

Shifted coordinate systems:



translate origin by $\underline{\Delta}(t)$

use vector addition

$\underline{r}' = \underline{r} - \underline{\Delta}(t)$ - shift observer

$\underline{v}' = \underline{v} - \dot{\underline{\Delta}}(t)$ - moving car

$\underline{a}' = \underline{a} - \ddot{\underline{\Delta}}(t)$ - falling elevator

Simple rule because each coordinate shifts independently

Rotations mix axes - more complicated - later.

Newton's Laws — derive from idealized experiments

N1 – "An isolated object moves at constant velocity"

(1) Nothing in nature is really isolated, so this is an extrapolation of lab experiments

(2) "Object" here could be anything, but for now focus on point particles, too small for their structure to be resolved.

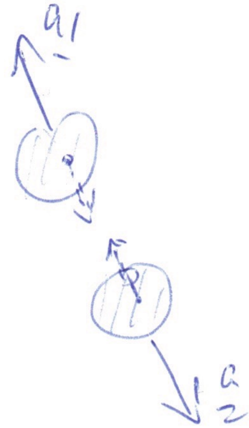
(3) Not true in general!

If you see velocity $\underline{v}$ = constant, another observer moving at $\underline{V}(t)$ with respect to you sees velocity $\underline{v} - \underline{V}$, and if $\underline{V} \neq$ constant ...

Better: For a certain class of observers, isolated objects move with constant velocity. Any two such observers have constant relative velocity.

Since observer ↔ reference frame, these are called "inertial reference frames"

N2 "Experiment" shows that isolated pairs of objects move towards or away from each other



$$\hat{a}_1 = -\hat{a}_2 \quad \text{and} \quad \frac{a_1}{a_2} = \mu_{21} = \text{constant}$$

μ_{21} is independent of where they are or how fast they move.

(exceptions - magnetism, relativity, QM)

Likewise, if 1 + 3 are isolated $\hat{a}_1 = -\hat{a}_3 \quad \frac{a_1}{a_3} = \mu_{31}$

if 2 + 3 " $\hat{a}_2 = -\hat{a}_3$ and $\frac{a_2}{a_3} = \mu_{32}$

and $\frac{\mu_{31}}{\mu_{21}} = \mu_{32}$, independent of position, speed, ...

So let #1 = "reference mass" = 1 kg or 1 lb or ...

and let $\mu_{21} \equiv m_2$, $\mu_{31} = m_3$ ... = "mass" of 2, 3, ...

Then $\frac{a_2}{a_3} = \mu_{32} = \frac{\mu_{31}}{\mu_{32}} = \frac{m_3}{m_2}$ or $m_2 a_2 = m_3 a_3$

add $\hat{a}_2 = -\hat{a}_3$: $m_2 \underline{a}_2 = -m_3 \underline{a}_3$

let $\boxed{\underline{F} = m\underline{a}}$ N2 then $\underline{F}_{23} = -\underline{F}_{32}$: N3

↑ force on 2 from 3

Simple 1-d examples:

(1) $F = F_0 \sin(\omega t + \varphi)$ a $f(t)$ – just integrate

$$m \frac{d^2x}{dt^2} = F_0 \sin(\omega t + \varphi)$$

$$m \frac{dx}{dt} = -\frac{F_0}{\omega} \cos(\omega t + \varphi) + c_1$$

$$m x = -\frac{F_0}{\omega^2} \sin(\omega t + \varphi) + c_1 t + c_2$$

$c_{1,2}$ = integration constants $\leftrightarrow$ $x(0)$ and $\dot{x}(0)$ or …

(2) $F = -k/x^3$ $k > 0$ a $f(x)$ – needs a trick

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$ (works in 1-d only)

$$\rightarrow \quad mv\frac{dv}{dx} = -\frac{k}{x^3} \quad \rightarrow \quad \frac{1}{2}mv^2 = \frac{k}{2x^2} + c_1$$

Suppose $x = x_0$ and $v = 0$ at $t = 0$; then $c_1 = -k/2x_0^2$

so $$v = \frac{dx}{dt} = \pm\sqrt{\frac{k}{m}\left(1/x^2 - 1/x_0^2\right)} \rightarrow -\sqrt{\frac{k}{mx_0^2}} \cdot \frac{\sqrt{x_0^2 - x^2}}{x}$$

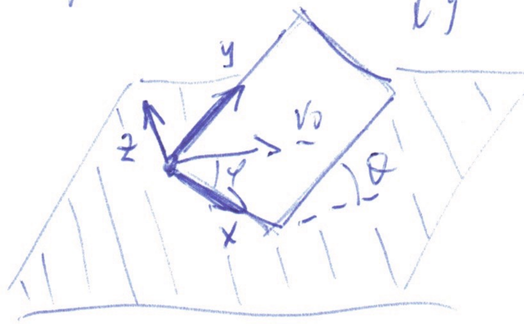
starts with $v = 0$ where $F < 0$: $v < 0$

or $$\underbrace{\int \frac{x\,dx}{\sqrt{x_0^2 - x^2}}}_{= -\sqrt{x_0^2 - x^2}} = -\sqrt{\frac{k}{2mx_0^2}}\, t + c_2 \rightarrow 0$$

$$\Rightarrow x_0^2 - x^2 = \frac{k}{mx_0^2} t^2 .$$

3-6 example

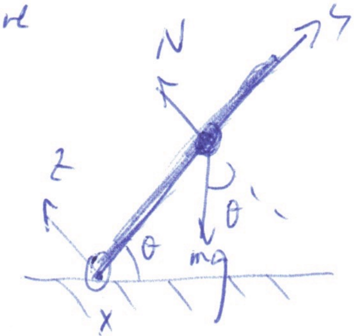
L9



A puck is thrown up a sheet of ice tilted at θ, at velocity $\underline{v}_0$. Gravity acts, no friction. Motion? Return point?

choose sheet = x-y plane

Look along $\hat{x}$:



$g_x = 0$

$g_y = -g\sin\theta$

$g_z = -g\cos\theta$

$m\ddot{\underline{r}} = \underline{F}$

$m\ddot{x} = 0$

$m\ddot{y} = -mg\sin\theta$

$m\ddot{z} = -mg\cos\theta + N \equiv 0$ because the puck stays on the ice

$\xrightarrow{\text{integrate}}$

$\dot{x} = \dot{x}_0 = v_0\cos\varphi$

$\dot{y} = \dot{y}_0 - \cancel{m}g\sin\theta\, t = v_0\sin\varphi - mg\sin\theta\, t$

$\xrightarrow{\text{integrate}}$

$$\left.\begin{aligned} x &= v_0\cos\varphi\, t \\ y &= v_0\sin\varphi\, t - \tfrac{1}{2} g\sin\theta\, t^2 \end{aligned}\right\}\ \text{motion in the sheet}$$

Puck returns to $y = 0$ at $t = \dfrac{2v_0\sin\varphi}{g\sin\theta}$, $x = \dfrac{v_0\sin 2\varphi}{g\sin\theta}$

$\underline{\text{Strategy}}$: sketch $\rightarrow$ "free body diagram" (coordinate system) $\rightarrow$ $\underline{F} = m\underline{a}$ $\rightarrow$ motion

So far: point masses What about finite-sized objects?

Object = collection of mass points ("atoms")

or set of grains in a rock or metal

or mathematical subregions of a material as in calculus

$\rightarrow m_1$ at $\underline{r}_1(t)$, m_2 at $\underline{r}_2(t)$ ... m_N at $\underline{r}_N(t)$

Total force on the object is

$$\underline{F} = \sum_{i=1}^{N} \underline{F}_i = \sum_{i=1}^{N} \underline{F}_i^{\,ext} + {\sum_{i,j=1}^{N}}' \underline{F}_{ij}$$

$\uparrow$ gravity, E+M, walls, ... $\uparrow$ force on i from j

$$\equiv \underline{F}^{ext} + \underline{F}^{int}$$

note $\underline{F}_{ii} = 0$

$$\underline{F}_{int} = {\sum_{i,j=1}^{N}}' \underline{F}_{ij}$$

$$\begin{aligned} = \; & 0 + \cancel{\underline{F}_{12}} + \cancel{\underline{F}_{13}} + \cdots + \cancel{\underline{F}_{1N}} \\ & + \cancel{\underline{F}_{21}} + \underline{F}_{22} + \cdots + F_{2N} \\ & + \cancel{\underline{F}_{31}} + \cdots \\ & \quad \vdots \\ & + \cancel{\underline{F}_{N1}} + \cdots + \underline{F}_{N,N-1} \end{aligned}$$

cancels in pairs because $\underline{F}_{ij} = -\underline{F}_{ji}$

Formal argument:

$$\underline{F}_{int} = {\sum_{ij=1}^{N}}' \underline{F}_{ij} = {\sum_{ji=1}^{N}}' \underline{F}_{ji}$$

: interchange dummy indices i and j

$$= {\sum_{ij=1}^{N}}' \left(-\underline{F}_{ij}\right) = -\underline{F}_{int} \quad \text{so} \quad \underline{F}_{int} = 0$$

Now use $\underline{F}_i = m\underline{a}_i$:

$$\underline{F} = \sum_i m_i \frac{d^2\underline{r}_i}{dt^2} = \frac{d^2}{dt^2}\sum_i m_i\underline{r}_i = \left(\sum_i m_i\right)\cdot\frac{d^2}{dt^2}\frac{\sum_i m_i\underline{r}_i}{\sum_i m_i}$$

define $M = \sum_i m_i$ = mass of the object

$\underline{R}(t) = \sum_i m_i\underline{r}_i / \sum_i m_i$ = center of mass

then $\underline{F} = \underline{F}^{ext} = M\frac{d^2\underline{R}}{dt^2}$ which is N2 for an object

NB: the individual $\underline{r}_i$ can be anything here, so the object need not be rigid.

Rest of Newton's Laws:

N1 : $\frac{d\underline{R}}{dt} = \underline{V}_{cm}$ = constant when $\underline{F}^{ext} = 0$

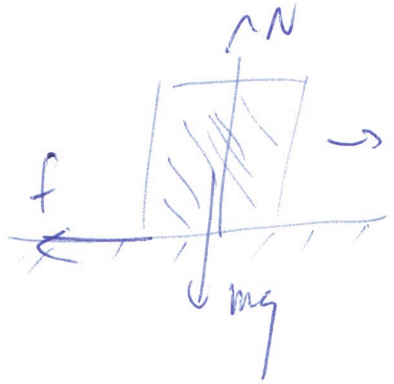
N3 : if objects 2 and 3 interact

$$\underline{F}_1 = \underline{F}_1^{ext} = \sum_{i\in 1}\underline{F}_i^{ext} = \sum_{i\in 1}\sum_{j\in 2}\underline{F}_{ij}$$

$$= \sum_{j\in 2}\sum_{i\in 1}\left(-\underline{F}_{ji}\right) = -\sum_{j\in 2}\underline{F}_j^{ext} = -\underline{F}_2$$

point masses satisfy N3

Solid-on solid friction:



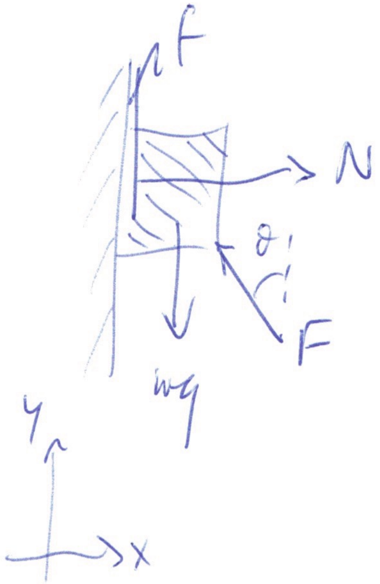
"laws"
$$\begin{cases} f = \mu_d N & \text{when moving} \\ f \le \mu_s N & \text{when at rest} \\ \hat{f} = \text{opposite motion} \end{cases}$$

$\mu_s, \mu_d < \mu_s$ = material constants

1-d motion: $m\ddot{x} = \sum(\text{other forces}) \pm \mu_d N$

1-d statics: $f = -\sum(\text{other forces})$ provided this is $\le \mu_s N$

Example: force needed to hold a block against a wall



$$F_x = N - F\sin\theta = 0$$

$$F_y = -mg + F\cos\theta + f = 0$$

so $f = mg - F\cos\theta \le \mu_s N = \mu_s F \cos\theta$

or $F \ge \dfrac{mg}{\cos\theta + \mu_s \sin\theta} \equiv F_{no\text{-}fall}$

limits: $\theta = 0$ "no contact" $F_{no\,fall} = mg$

$\theta = \frac{\pi}{2}$ $N = F$ $F_{no\text{-}fall} = mg/\mu_s$

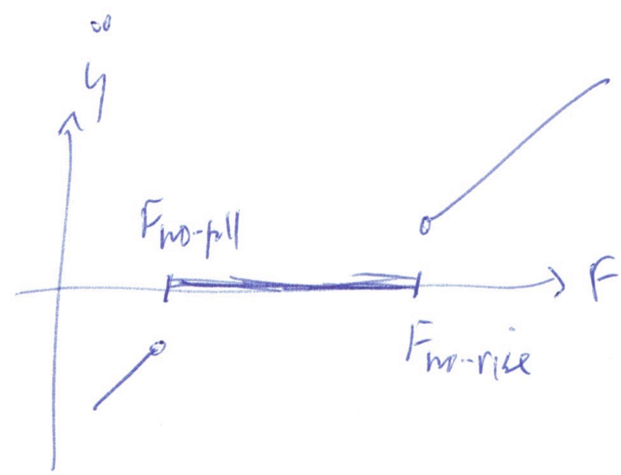
What's the minimum F to push the block $\underline{\underline{\text{up}}}$ the wall?

motion reverses → f changes sign

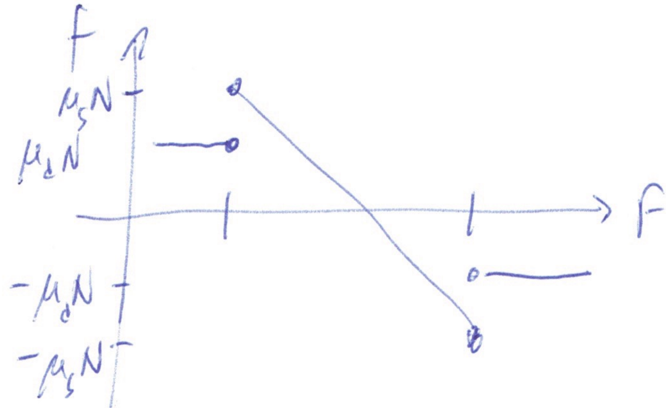
$$F_y = -mg + F\cos\theta - \mu_s F\sin\theta = 0$$

$$F_{no\text{-}rise} = \frac{mg}{\cos\theta - \mu_s \sin\theta} > F_{no\text{-}fall}$$

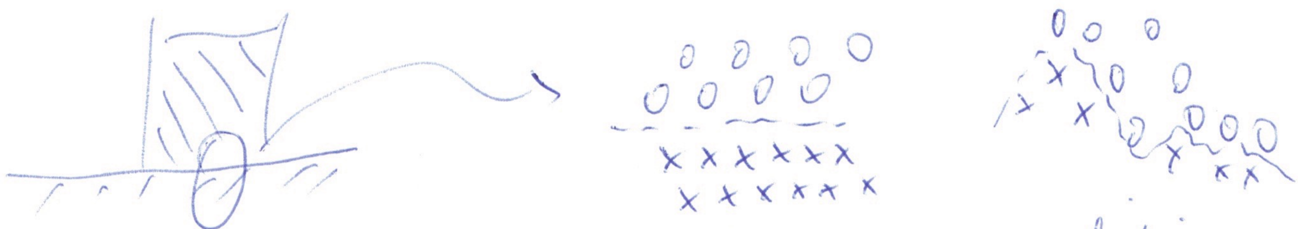
For $F_{no\text{-}fall} < F < F_{no\text{-}rise}$ static friction holds the block in place



$\ddot{y}$ jumps to a positive value because $\mu_k < \mu_s$



Origin of solid friction



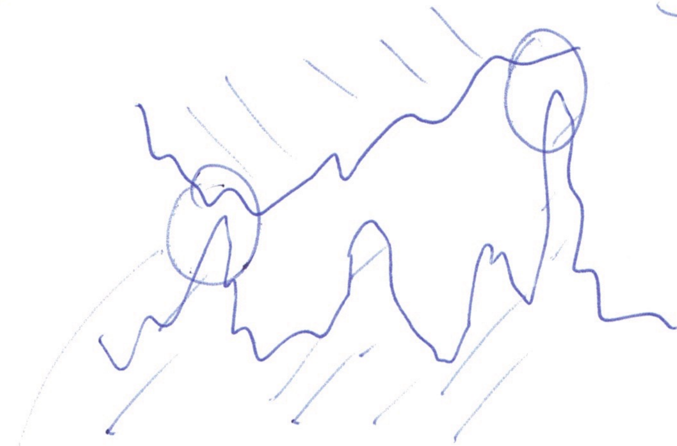
continuum scale

ideal

realistic

atomic scale

intermediate → scale



→ most resistance from close contacts

force $\propto$ contact area
contact area $\propto N$
$\Big\}$ $F \propto N$

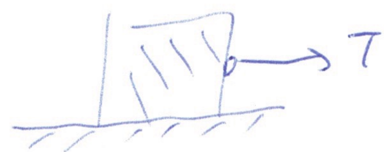
Statics: need to push protrusions "out of the way"

Dynamics: big protrusions pushed away or ground down

$\rightarrow \mu_s > \mu_d$

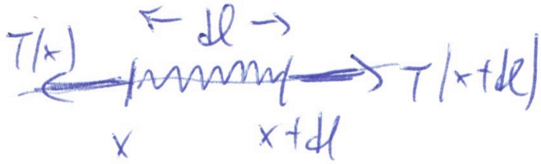
Real life: μ_s increases with time (aging)
μ_d decrease " " (smoothing)
sharp transition → stick/slip motion
interstitial lubricant film common

Curvature effects: pull a mass along a surface with a string

flat case: 

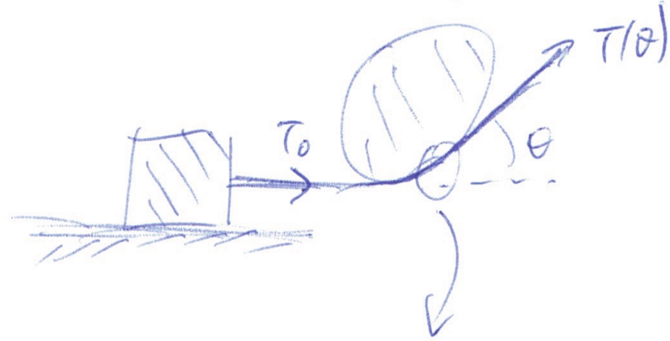
tension $T \geq T_0 = \mu_s mg$ to move

NB - T = constant if the string is massless

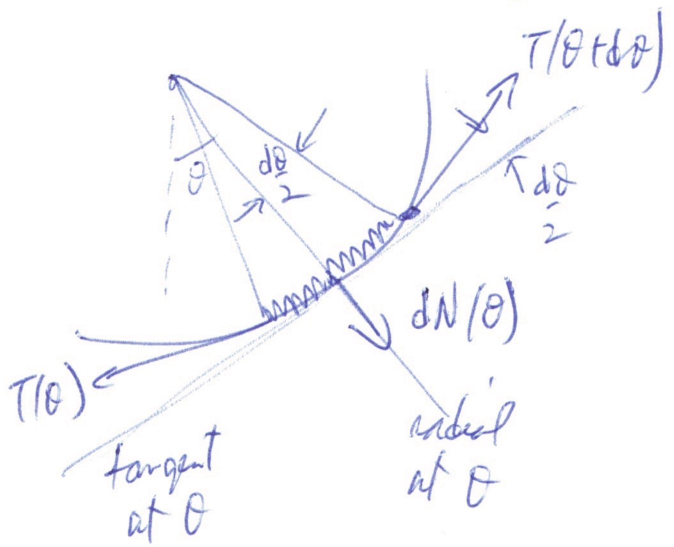


$$T(x+dl) - T(x) = (\lambda\, dl)\,\ddot{x} \underset{\lambda \to 0}{\longrightarrow} 0$$

curved case: wrap the string around a cylinder



claim $T(\theta) > T_0$



radial force balance

$$dN(\theta) = T(\theta) \sin\frac{d\theta}{2} + T(\theta + d\theta) \sin\frac{d\theta}{2}$$

tangential force balance

$$T(\theta+d\theta)\cos\frac{d\theta}{2} = T(\theta)\cos\frac{d\theta}{2} + \mu_s\, dN(\theta)$$

use $\sin\frac{d\theta}{2} \simeq \frac{d\theta}{2}$, $\cos\frac{d\theta}{2} \simeq 1$, $T(\theta+d\theta) = T(\theta) + T'(\theta)\,d\theta$

$\rightarrow dN(\theta) = T(\theta)\,d\theta$, $T'(\theta) = \mu_s T(\theta) \rightarrow$

$$T(\theta) = T(0)\, e^{\mu_s \theta} \quad !$$