Physics 71100 **Analytical Dynamics** **Fall 2014**

Final Examination – December 19
Do any four of the five problems – 50 points each – 200 points total

1. An observer in an inertial reference frame sees two relativistic particles moving with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$. Show that the magnitude of the *relative* velocity of the particles is given by
$$\mathbf{V}^2 = \frac{(\mathbf{u}_1 - \mathbf{u}_2)^2 - (\mathbf{u}_1 \times \mathbf{u}_2)^2/c^2}{(1 - \mathbf{u}_1 \cdot \mathbf{u}_2/c^2)^2}$$

2. A ball bounces elastically from the floor of an elevator at rest. How does the maximum height of the ball change when
(a) The elevator accelerates slowly (compared to g)?
(b) The elevator moves slowly (compared to the speed of the ball) with a constant velocity?

3. Given the Lagrangian
$$L = e^{\lambda t/m}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right)$$
where λ, m, and ω are constants,
(a) Find the Hamiltonian.
(b) Using the canonical transformation $F_2 = qPe^{\lambda t/2m}$, where P is the new momentum, find the new Hamiltonian K. (See the table on the next page.)
(c) Evaluate the Poisson brackets $[Q, K]$ and $[P, K]$, where Q is the new coordinate.
(d) Using the results of part (c), find $q(t)$.

4. A particle of mass m moves in one dimension in the potential
$$U(x) = \frac{a}{\sin^2(x/x_0)} \qquad a, x_0 > 0$$
Write down and solve the Hamilton-Jacobi equation. Find the conditions under which action-angle variables can be introduced, and use them to find the frequency of oscillation in that case. Note: $\int dx/\sqrt{1-x^2} = \sin^{-1} x$.

5. Consider the Lagrangian density
$$\mathcal{L} = \frac{1}{2}\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t} + \frac{\alpha}{6}\left(\frac{\partial\psi}{\partial x}\right)^3 - \frac{\nu}{2}\left(\frac{\partial^2\psi}{\partial x^2}\right)^2$$
for a scalar field $\psi(x, t)$, and assume that $\psi \to 0$ as $|x| \to \infty$. Find the Euler-Lagrange equation of motion, and show that $\phi(x, t) \equiv \partial\psi/\partial x$ satisfies the "Korteweg-deVries equation" which describes solitons:
$$\frac{\partial\phi}{\partial t} + \alpha\phi\frac{\partial\phi}{\partial x} + \nu\frac{\partial^3\phi}{\partial x^3} = 0$$
Explain in particular how you treat the term in $\mathcal{L}$ involving $\partial^2\psi/\partial x^2$.

TABLE 9.1 Properties of the Four Basic Canonical Transformations

Generating Function	Generating Function Derivatives	Trivial Special Case
$F = F_1(q, Q, t)$	$p_i = \dfrac{\partial F_1}{\partial q_i} \qquad P_i = -\dfrac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i, \quad Q_i = p_i, \quad P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \dfrac{\partial F_2}{\partial q_i} \qquad Q_i = \dfrac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i, \quad Q_i = q_i, \quad P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\dfrac{\partial F_3}{\partial p_i} \qquad P_i = -\dfrac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i, \quad Q_i = -q_i, \quad P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\dfrac{\partial F_4}{\partial p_i} \qquad Q_i = \dfrac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i, \quad Q_i = p_i, \quad P_i = -q_i$