Non-holonomic Constraints

In class we discussed the Lagrangian treatment of holonomic ($b_j(\mathbf{q})$=0) and linear non-holonomic ($\sum_i a_{ji}(\mathbf{q})dq_i$=0) constraints. Note that the time derivative of the holonomic constraint automatically takes the form of a linear non-holonomic one, with $a_{ji} = \partial b_j/\partial q_i$. We arrived at the equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_j \lambda_j a_{ji} \qquad i = 1 \ldots n \tag{1}$$

Goldstein's book goes further, and states that the equation of motion for general velocity-dependent constraints of the form $f_j(q_i, \dot{q}_i) = 0$ can be found by extremizing $\int dt(L+\sum_j \lambda_j f_j)$ with respect to q_i and λ_j using the usual calculus of variation rules, leading to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_j \lambda_j \frac{\partial f_j}{\partial q_i} - \frac{d}{dt}\left(\lambda_j \frac{\partial f_j}{\partial \dot{q}_i}\right) \tag{2}$$

If f_j depends only on $\mathbf{q}$ then Eqs. (1) and (2) agree, but when $f_j = \sum_i a_{ji}(q)\dot{q}_i$ is linear in velocity, they disagree. Since (1) is equivalent to $\mathbf{F} = m\mathbf{a}$, it is certainly correct, so (2) must be wrong. A particular problem with (2) is that it yields a differential equation for $\lambda_j(t)$ which requires an initial condition, but there is no obvious way of determining it.

For discussions of the discrepancy, see
J. R. Ray, Amer. J. Phys. **34**, 406 (1966) – incorrect paper
J. R. Ray, *ibid.* 1202 – retraction
I. R. Gatland, *ibid.* **72**, 941 (2004) – test case
M. R. Flannery, *ibid.* **73**, 265 (2005) – general discussion

Treatments of more general non-holonomic constraints may be found in
L. A. Pars, *Treatise on Analytical Mechanics* (Oxbow Press, 1972)
F. E. Udwadia and R. E. Kalabia *Analytical Dynamics* (Cambridge 1996).

This subject is very murky in fact; a common starting point is "Gauss' principle of least constraint" which involves taking the derivative of the constraint equation which then involves the accelerations, and it is not clear if the result is correct. The derivation is as folows: suppose a system of particles is at $\{\mathbf{r}_i(t), \mathbf{p}_i(t) = m_i\mathbf{v}_i(t)\}$ at time t, so its Newtonian evolution is

$$\delta\mathbf{r}_i = \mathbf{v}_i(t)dt + \frac{1}{2}\mathbf{a}_i(t)dt^2$$

where $\mathbf{a}_i \equiv \ddot{\mathbf{r}}_i$. If $\delta\tilde{\mathbf{r}}_i$ is any other displacement consistent with the constraints but not necessarily given by Newton's law, then

$$\delta\tilde{\mathbf{r}}_i = \mathbf{v}_i(t)dt + \frac{1}{2}\tilde{\mathbf{a}}_i(t)dt^2$$

with the *same* $\mathbf{v}_i$ because it is part of the specification of the system at time t. Therefore,

$$\delta(\mathbf{r}_i - \tilde{\mathbf{r}}_i) = \frac{1}{2}(\mathbf{a}_i - \tilde{\mathbf{a}}_i)dt^2$$

Now d'Alembert's principle of virtual work states that

$$\sum_i (\mathbf{F}_i - m_i \mathbf{a}_i) \cdot \delta \mathbf{r}_i = 0 \tag{3}$$

where $\mathbf{F}_i$ is the sum of all non-constraint forces and $\delta \mathbf{r}_i$ is any displacement consistent with the constraints. Applying this equation to $\delta(\mathbf{r}_i - \tilde{\mathbf{r}}_i)$,

$$\sum_i (\mathbf{F}_i - m_i \mathbf{a}_i) \cdot \delta \mathbf{a}_i = 0$$

Now $\delta \mathbf{F}_i = 0$, because the non-constraint force is completely determined by $\mathbf{r}_i$ and $\mathbf{v}_i$, so

$$\sum_i (\mathbf{F}_i - m_i \mathbf{a}_i) \cdot \delta(\mathbf{F}_i - m_i \mathbf{a}_i) = 0$$

where the variation is over the $\mathbf{a}_i$, or

$$\delta \sum_i \frac{(\mathbf{F}_i - m_i \mathbf{a}_i)^2}{2m_i} = 0 \tag{4}$$

This is the principle of "least constraint." To use it, write the constraint equations in the form $g_j(\mathbf{r}_i, \dot{\mathbf{r}}_i, \ddot{\mathbf{r}}_i) = 0$ by differentiating the f_j and extremize

$$G = \sum_i \frac{(\mathbf{F}_i - m_i \mathbf{a}_i)^2}{2m_i} + \sum_j \lambda_j g_j$$

with respect to $\ddot{\mathbf{r}}_i$ and the Lagrange multipliers λ_j, which gives

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_j \lambda_j \frac{\partial g_j}{\partial \ddot{\mathbf{r}}_i} \tag{5}$$

One can verify that (5) gives the standard result for simple examples with holonomic and linear non-holonomic constraints, but in general it may not agree with Newton, so its status is uncertain.