Second Exam Solutions

1. The relativistic force law is $F^\mu = \frac{dp^\mu}{d\tau}$ where τ = proper time, $p^\mu = (E/c, \gamma \underline{u})$,

$$F^\mu = \left(\frac{\gamma}{c} \underline{u}\cdot\underline{F},\ \underline{F} + (\gamma-1)\frac{1}{u^2}\,\underline{u}\cdot\underline{F} \right)$$

& $\underline{F}$ is the force in the rest frame. Here, $\underline{u}$ & $\underline{F}$ are along $\hat{x}$

So $\gamma F = \frac{d}{d\tau}(\gamma u) \xrightarrow[dt=\gamma d\tau]{} F = \frac{d}{dt}(\gamma u)$

or $-w^2 x = \dot{\gamma} u + \gamma \dot{u} = \gamma^3 \dot{u}$

To solve, multiply both sides by $\dot{x} = u$

$$-w^2 x\dot{x} = \gamma^3 u\dot{u} = c^2 \frac{d\gamma}{dt}$$

or $$-\frac{1}{2}w^2 x^2 = c^2\gamma + K \qquad K = \text{constant}$$

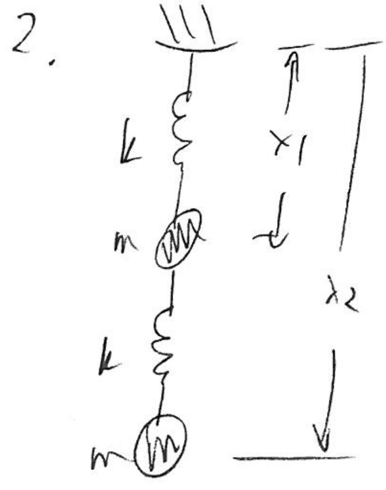
If the particle is at rest at $x = a$, $K = -\frac{1}{2}w^2 a^2 - c^2$

and $$\gamma = 1 + \frac{w^2}{2c^2}(a^2 - x^2)$$

& $$u = c\sqrt{1 - \left(1 + \frac{w^2}{2c^2}(a^2-x^2)\right)^2}$$

For $|wa| \ll c$ $\quad u \to w\sqrt{a^2 - x^2}$ = NR result.

2.



$$L = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2} k x_1^2 - \frac{1}{2} k (x_1 - x_2)^2$$

where the x_i are displacements from equilibrium

$$L = \frac{1}{2} \underline{\dot{x}}\, \underline{\underline{T}}\, \underline{\dot{x}} - \frac{1}{2} \underline{x}\, \underline{\underline{V}}\, \underline{x}$$

with $\underline{\underline{T}} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ $\underline{\underline{V}} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}$

So the normal mode frequencies satisfy

$$\det\left(\omega^2 \underline{\underline{T}} - \underline{\underline{V}}\right) = 0 = (m\omega^2 - 2k)(m\omega^2 - k) - k^2$$

or $\omega^4 m^2 - 3k\omega^2 m + k^2 = 0$

$$\rightarrow \quad \omega_{1,2} = \frac{k}{m}\left[\frac{3}{2} \pm \frac{1}{2}\sqrt{5}\right]$$

The modes are given by

$$\frac{k}{2m}(3 \pm \sqrt{5}) \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} x_{1\pm} \\ x_{2\pm} \end{pmatrix} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} x_{1\pm} \\ x_{2\pm} \end{pmatrix}$$

or $\dfrac{x_{2\pm}}{x_{1\pm}} = \dfrac{1 \pm \sqrt{5}}{2}$, $\underline{y}_\pm = \dfrac{1}{\sqrt{x_{1\pm}^2 + x_{2\pm}^2}} \begin{pmatrix} x_{1\pm} \\ x_{2\pm} \end{pmatrix}$

3. The transformation $(q, p) \to (\underline{Q}, \underline{P})$ is canonical

if $\quad [Q_i, P_j]_{(q,p)} = \delta_{ij}$

or $\displaystyle \sum_k \frac{\partial Q_i}{\partial q_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k}\frac{\partial P_j}{\partial q_k} \to \sum_k \underbrace{\frac{\partial Q_i}{\partial q_k}}_{\equiv m_{ik}}\frac{\partial P_j}{\partial p_k} = \delta_{ij}$

Apply $\sum_i m^{-1}_{\ell i}$ to both sides:

$$\frac{\partial P_j}{\partial p_\ell} = (m^{-1})_{\ell j} \to P_j = \sum_\ell (m^{-1})_{\ell j}\, p_\ell$$

This is the same result as letting $L(q, \dot q) \to L(\underline{Q}(q), \dot{\underline{Q}}(q))$

& defining $\displaystyle P_j = \frac{\partial L}{\partial \dot Q_j} = \sum_i \frac{\partial L}{\partial \dot q_i}\frac{\partial \dot q_i}{\partial \dot Q_j} = \sum_i p_i (m^{-1})_{ij}$,

since $\displaystyle \dot q_i = \sum_k \frac{\partial q_i}{\partial Q_k}\dot Q_k$ & $\displaystyle \frac{\partial \dot q_i}{\partial \dot Q_j} = \frac{\partial q_i}{\partial Q_j} = (m^{-1})_{ij}$

Alternately use $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$ as the condition, where

$$\underline{\underline{M}} = \begin{pmatrix} \partial \underline{Q}/\partial q & 0 \\ \partial \underline{P}/\partial q & \partial \underline{P}/\partial p \end{pmatrix} \text{ here}$$