First Exam Solutions

3. $$L = \sum_{i=1}^{N} \frac{1}{2} m \dot{\underline{r}}_i^2 - \sum_{i<j} V_2(\underline{r}_i - \underline{r}_j)$$

constraint: $\frac{1}{N} \sum \dot{\underline{r}}_i = \underline{U}(t) \rightarrow \sum_1^N d\underline{r}_i - N\underline{U}\,dt = 0$

Method: if $L(\underline{q}, \dot{\underline{q}}, t)$ has k constraints of the form

$\sum_i a_{ji}(t)\, dq_i = 0, \; j = 1, 2, \ldots, k$ then the eqs of motion are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{k} \lambda_j a_{ji}(t).$$

So here
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\underline{r}}_i} - \frac{\partial L}{\partial \underline{r}_i} = m \ddot{\underline{r}}_i - \sum_{j \neq i} \underline{F}_{ij} = \underline{\lambda}$$

where $\underline{F}_{ij} = -\frac{\partial}{\partial \underline{r}_i} V_2(\underline{r}_i - \underline{r}_j)$

To find $\underline{\lambda}$, add the eqs:

$$\sum m \ddot{\underline{r}}_i + 0 = N \underline{\lambda} \qquad (\text{using } \underline{F}_{ij} = -\underline{F}_{ji})$$

or $\underline{\lambda} = m \dot{\underline{U}}(t)$

$$\Rightarrow \quad m \ddot{\underline{r}}_i = \sum_j \underline{F}_{ij} + m \dot{\underline{U}}(t)$$

Check: add the eqs. again: $\sum m \ddot{\underline{r}}_i = m N \dot{\underline{U}}$

or $\frac{d}{dt}\left(\frac{1}{N} \sum \dot{\underline{r}}_i - \underline{U}\right) = 0$; if constraint imposed at $t = 0$ then it holds for all t.

1. In spherical coordinates, the element of arc length is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

so on the surface of a sphere ($r = 1$ say), if θ & φ are written as functions of a parameter λ then

$$D_{12} = \int_1^2 ds = \int_1^2 d\lambda \sqrt{\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta \left(\frac{d\varphi}{d\lambda}\right)^2}$$

The Euler - Lagrange eqs →

$$\frac{d}{d\lambda}\left(\frac{\theta'(\lambda)}{\sqrt{\theta'^2 + \sin^2\theta\, \varphi'^2}}\right) = \frac{\sin\theta \cos\theta\, \varphi'^2}{\sqrt{\cdots}}$$

$$\frac{d}{d\lambda}\left(\frac{\sin^2\theta\, \varphi'(\lambda)}{\sqrt{\cdots}}\right) = 0$$

Without loss of generality the 2 points can be chosen as $(\theta = 0, \varphi = 0)$ & $(\theta = \theta_0, \varphi = 0)$ & the solution of the 2 ODE's subject to these initial conditions is

$$\theta(\lambda) = \lambda\theta_0, \quad 0 \le \lambda \le 1, \quad \varphi(\lambda) = 0$$

which is part of a great circle.

↓

Method 2: write the curve as $\underline{r} = \underline{r}(t)$ t = label of points

then $ds = \sqrt{d\underline{r}^2(t)} = \sqrt{\underline{r}'(t)^2}\, dt$ & the

problem is to minimize $\int ds$ subject to $\underline{r}^2 = 1$ say.

$$\rightarrow \quad \delta \int dt \left[\sqrt{\underline{r}'(t)^2} + \lambda\left(\underline{r}^2(t) - 1\right) \right] = 0$$

or $$\frac{d}{dt}\left(\frac{\underline{r}'(t)}{\sqrt{\underline{r}'^2(t)}} \right) = \frac{d}{dt}\left(\hat{r}'(t) \right) = 2\lambda\, \underline{r}(t)$$

Take $\hat{r}(t) \times$ this equation:

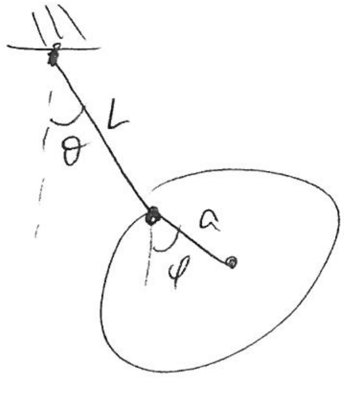
$$\frac{d}{dt}\left(\hat{r} \times \hat{r}' \right) = 0 \quad \text{so} \quad \hat{r} \times \hat{r}' = \underline{K} = \text{constant vector.}$$

Also $\underline{r}^2 = 1 \rightarrow \underline{r} \cdot \underline{r}' = 0 = \hat{r} \cdot \hat{r}'$

So $\hat{r}'(t)$ is tangent to the sphere, and $\perp$ to a fixed space vector $\underline{K}$, which is only possible if $\hat{r}'$ moves on a great circle

e.g. Start at the north pole & assume $\hat{K}$ = east. Then $\hat{r}'$ must point south at $t = 0$, and if it continues south it maintains $\hat{r} \times \hat{r}' = \underline{K}$.

2.



$V = V_{rod} + V_{disc}$

$= -\frac{mgL}{2}\cos\theta - mg(L\cos\theta + a\cos\varphi)$

$T = T_{rod} + T_{disc,cm} + T_{disc,rel}$

$T_{rod} = \frac{1}{2} I_r \dot\theta^2$, I_r = moment of inertia about an end of the rod

$T_{disc,cm} = \frac{m}{2}(\dot x_c^2 + \dot y_c^2)$ where $\begin{cases} x_c = L\sin\theta + a\sin\varphi \\ y_c = L\cos\theta + a\cos\varphi \end{cases}$

$= \frac{m}{2}\left(L^2\dot\theta^2 + a^2\dot\varphi^2 + 2aL\dot\theta\dot\varphi\cos(\theta-\varphi)\right)$

$T_{disc,rel} = \frac{1}{2} I_d \dot\varphi^2$, I_d = moment of inertia of disc about center

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} - \frac{\partial \mathcal{L}}{\partial \theta} = 0 = \frac{d}{dt}\left(I_r\dot\theta + mL^2\dot\theta + maL\dot\varphi\cos(\theta-\varphi)\right)$$

$$+ maL\dot\theta\dot\varphi\sin(\theta-\varphi) + \frac{mgL}{2}\sin\theta + mgL\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\varphi} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 = \frac{d}{dt}\left(ma^2\dot\varphi + I_d\dot\varphi + maL\dot\theta\cos(\theta-\varphi)\right)$$

$$+ maL\dot\theta\dot\varphi\sin(\theta-\varphi) + mga\sin\varphi$$

or

$$0 = (I_r + mL^2)\ddot\theta + maL\ddot\varphi\cos(\theta-\varphi) - maL\dot\varphi^2\sin(\theta-\varphi) + \frac{3}{2}mgL\sin\theta$$

$$0 = (ma^2 + I_d)\ddot\varphi + maL\ddot\theta\cos(\theta-\varphi) - maL\dot\theta^2\sin(\theta-\varphi) + mga\sin\varphi$$

4. $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \quad \to$

$$m\ddot{r} = -mr\dot{\theta}^2 - V'(r) \quad \text{and} \quad l \equiv mr^2\dot{\theta} = \text{constant}$$

so $$m\ddot{r} = -\frac{l^2}{mr^3} - V'(r).$$

Also $$E = \frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) \equiv \frac{m}{2}\dot{r}^2 + V_{eff}(r) = \text{constant}$$

A stable circular orbit at $r = R$ requires a minimum V_{eff}:

$$\frac{\partial V_{eff}}{\partial r} = 0 \quad \text{and} \quad \frac{\partial^2 V_{eff}}{\partial r^2} > 0 \quad \text{at } r = R$$

$$\to \quad V'(R) = \frac{l^2}{mR^3}, \quad V''(R) + \frac{3l^2}{mR^4} = V''(R) + \frac{3}{R}V'(R) > 0$$

(Also $\dot{r} = 0$ at $r = R \to E = V_{eff}(R)$ which fixes E.)

For small oscillations write $r = R + x$, $|x| \ll R$, &

$$m\ddot{x} = -\frac{l^2}{m(R+x)^3} - V'(R+x)$$

$$= \cancel{-\frac{l^2}{mR^3}} + \frac{3l^2 x}{mR^4} \cancel{- V'(R)} - x V''(R) + O(x^2)$$

$$= -\left(V''(R) + \frac{3}{R}V'(R)\right)x \equiv -m\omega^2 x$$

Note $\omega^2 > 0$ as required. If $V(r) = -kr^n$ for example,

$$V''(R) + \frac{3}{R}V'(R) = -knR^{n-2}(n+2) \quad \text{so} \quad n > -2.$$